Ideal lowpass filtering: convolution with sinc(x) function

An ideal lowpass filter can be described as a zero-centred pulse function $F(\omega)$ in the frequency domain $\omega$: it removes all frequencies higher than some cut-off frequency $W$. For simplicity we will initially set $W = \pm 1$, and give the pulse function unit area: $F(\omega) = 1/2$ for $\omega \in [-1, +1]$, and $F(\omega) = 0$ for $|\omega| > 1$:

$$F(\omega) = \begin{cases} 
1/2 & \text{for } \omega \in [-1, +1] \\
0 & \text{for } |\omega| > 1
\end{cases}$$

The inverse Fourier transform of this $F(\omega)$ lowpass filter is $f(x)$:

$$f(x) = \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} d\omega = \frac{1}{2} \int_{-1}^{+1} e^{i\omega x} d\omega$$

$$= \frac{1}{2} \left[ \frac{e^{i\omega x}}{ix} \right]_{\omega=-1}^{\omega=+1} = \frac{e^{ix} - e^{-ix}}{2ix} = \frac{\sin(x)}{x}.$$

(At $x = 0$, $f(x) = \frac{1}{2} \int_{-1}^{+1} d\omega = 1$.) This wiggly function is called sinc.
The zero-crossings are at \( x = n\pi \) for all (\( \pm \)) integers \( n \) except \( n = 0 \).

In the more general case when we may wish to remove from a signal all frequencies higher than \( W \) in Hertz, so the lowpass filter has cut-off at \( \pm W \), the corresponding time-domain sinc\((t)\) function is parameterised by that frequency \( W \), and it wiggles faster the higher \( W \) is:

\[
\frac{\sin(2\pi Wt)}{2\pi Wt}
\]