Further Graphics

**NURBS**

Non-Uniform Rational B-Splines

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NURBS curves

Like Bezier cubics, NURBS curves are parametric. Their shape is determined by:

- control points, $P_i$
- the *NURBS basis functions*, $N_{i,k}$

\[
P(t) = \sum_{i=1}^{n} N_{i,k}(t) P_i
\]
Properties of NURBS curves

1. The basis functions must sum to 1.0

\[ P(t) = \sum_{i=1}^{n} N_{i,k}(t)P_i \]

\[ \sum_{i=1}^{n} N_{i,k}(t) = 1, \quad t_{\text{min}} \leq t \leq t_{\text{max}} \]
Properties of NURBS curves

2. The basis functions are calculated from a knot vector
   - This is a non-decreasing sequence of real numbers
     - e.g. [0,0,0,1,1,1]
     - or [1,2,3,4,5,6]
     - or [1.2, 3.4, 5.6, 5.6, 7.2, 15.6]

\[ P(t) = \sum_{i=1}^{n} N_{i,k}(t)P_i \]
Properties of NURBS curves

3. If the basis functions are $C^m$-continuous at $t$, then $P(t)$ is guaranteed to be $C^m$-continuous at $t$
   - So continuity depends only on the basis functions, $N_{i,k}$
     - Continuity does not depend on the locations of the control points

$$P(t) = \sum_{i=1}^{n} N_{i,k}(t)P_i$$
Properties of NURBS surfaces

NURBS surfaces are a bivariate generalisation of the univariate NURBS curve

\[ P(t) = \sum_{i=1}^{n} N_{i,k}(t)P_i \]

\[ P(s,t) = \sum_{i=1}^{m} \sum_{j=1}^{n} N_{i,k}(s)N_{j,k}(t)P_{i,j} \]
NURBS

- **NURBS** (“Non-Uniform Rational B-Splines”) are a generalization of the Bezier curve concept:
  - NU: *Non-Uniform*. The knots in the knot vector are not required to be uniformly spaced.
  - R: *Rational*. The spline may be defined by rational polynomials (homogeneous coordinates.)
  - BS: *B-Spline*. A generalization of Bezier splines with controllable degree.
B-Splines

We’ll build our definition of a B-spline from:

- \( d \), the *degree* of the curve
- \( k = d+1 \), called the *parameter* of the curve
- \( \{P_1 \ldots P_n\} \), a list of *n control points*
- \([t_1, \ldots, t_{k+n}]\), a *knot vector* of \((k+n)\) parameter values ("knots")
- \( d = k-1 \) is the degree of the curve, so \( k \) is the number of control points which influence a single interval
  - Ex: a cubic \((d=3)\) has four control points \((k=4)\)
- There are \( k+n \) knots \( t_i \), and \( t_i \leq t_{i+1} \) for all \( t_i \)
- Each B-spline is \( C^{(k-2)} \) continuous:
  - *continuity* is degree minus one,
  - so a \( k=3 \) curve has \( d=2 \) and is \( C^1 \)

http://www.mikekrummhoefener.com/toy-story-char-grid/
B-Splines

- A B-spline curve is defined between $t_{min}$ and $t_{max}$:

\[ P(t) = \sum_{i=1}^{n} N_{i,k}(t) P_i, \quad t_{min} \leq t < t_{max} \]

- $N_{i,k}(t)$ is the *basis function* of control point $P_i$ for parameter $k$. $N_{i,k}(t)$ is defined recursively:

\[
N_{i,1}(t) = \begin{cases} 
1, & t_i \leq t < t_{i+1} \\
0, & \text{otherwise}
\end{cases}
\]

\[
N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)
\]
B-Splines

\begin{align*}
&k=1 \quad N_{1,1}(t) \quad N_{2,1}(t) \quad N_{3,1}(t) \quad N_{4,1}(t) \quad \ldots \\
&k=2 \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&\quad N_{1,2}(t) \quad N_{2,2}(t) \quad N_{3,2}(t) \quad \ldots \\
&k=3 \quad \downarrow \quad \downarrow \quad \downarrow \\
&\quad N_{1,3}(t) \quad N_{2,3}(t) \quad \ldots \\
&k=4 \quad \downarrow \\
&\quad N_{1,4}(t) \quad \ldots
\end{align*}
B-Splines

Knot vector = \{0, 1, 2, 3, 4, 5\}, \( k = 1 \rightarrow d = 0 \) (degree = zero)

\[ N_{i,1}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \]
B-Splines

\[ N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \]

Knot vector = \{0,1,2,3,4,5\}, \( k = 2 \rightarrow d = 1 \) (degree = one)
B-Splines

\[
N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)
\]

Knot vector = \{0,1,2,3,4,5\}, \ k = 3 \rightarrow d = 2 \ (degree = two)
Basis functions really sum to one (k=2)

The sum of the four basis functions is fully defined (sums to one) between $t_2$ (t=1.0) and $t_5$ (t=4.0).
Basis functions really sum to one (k=3)

The sum of the three functions is fully defined (sums to one) between \( t_3 \) (t=2.0) and \( t_4 \) (t=3.0).
B-Splines

At $k=2$ the function is piecewise linear, depends on $P_1, P_2, P_3, P_4$, and is fully defined on $[t_2, t_5)$.

At $k=3$ the function is piecewise quadratic, depends on $P_1, P_2, P_3$, and is fully defined on $[t_3, t_4)$.

Each parameter-$k$ basis function depends on $k+1$ knot values; $N_{i,k}$ depends on $t_i$ through $t_{i+k}$, inclusive. So six knots $\rightarrow$ five discontinuous functions $\rightarrow$ four piecewise linear interpolations $\rightarrow$ three quadratics, interpolating three control points. $n=3$ control points, $d=2$ degree, $k=3$ parameter, $n+k=6$ knots.

Knot vector = \{0,1,2,3,4,5\}
Non-Uniform B-Splines

- The knot vector \{0,1,2,3,4,5\} is uniform:
  \[ t_{i+1} - t_i = t_{i+2} - t_{i+1} \forall t_i. \]
- Varying the size of an interval changes the parametric-space distribution of the weights assigned to the control functions.
- Repeating a knot value reduces the continuity of the curve in the affected span by one degree.
- Repeating a knot \(k\) times will lead to a control function being influenced only by that knot value; the spline will pass through the corresponding control point with C0 continuity.
Open vs Closed

- A knot vector which repeats its first and last knot values $k$ times is called *open*, otherwise *closed*.
- Repeating the knots $k$ times is the only way to force the curve to pass through the first or last control point.
- Without this, the functions $N_{1,k}^l$ and $N_{n,k}^n$ which weight $P_1$ and $P_n$ would still be ‘ramping up’ and not yet equal to one at the first and last $t_i$. 
Open vs Closed

- Two examples you may recognize:
  - $k=3$, $n=3$ control points, knots=$\{0,0,0,1,1,1\}$
  - $k=4$, $n=4$ control points, knots=$\{0,0,0,0,1,1,1,1\}$
Non-Uniform *Rational B-Splines*

- Repeating knot values is a clumsy way to control the curve’s proximity to the control point.
  - We want to be able to slide the curve nearer or farther without losing continuity or introducing new control points.
- The solution: *homogeneous coordinates*.
- Associate a ‘weight’ with each control point: $\omega_i$. 

Non-Uniform Rational B-Splines

- Recall: \([x, y, z, \omega]_H \rightarrow [x / \omega, y / \omega, z / \omega]\)
- Or: \([x, y, z, 1] \rightarrow [x\omega, y\omega, z\omega, \omega]_H\)
- The control point
  \[P_i = (x_i, y_i, z_i)\]
  becomes the homogeneous control point
  \[P_{iH} = (x_i\omega, y_i\omega, z_i\omega)\]
- A NURBS in homogeneous coordinates is:
  \[P_H(t) = \sum_{i=1}^{n} N_{i,k}(t)P_{iH}, \quad t_{min} \leq t < t_{max}\]
Non-Uniform Rational B-Splines

- To convert from homogeneous coords to normal coordinates:

\[ x_H(t) = \sum_{i=1}^{n} (x_i \omega_i)(N_{i,k}(t)) \]
\[ y_H(t) = \sum_{i=1}^{n} (y_i \omega_i)(N_{i,k}(t)) \]
\[ z_H(t) = \sum_{i=1}^{n} (z_i \omega_i)(N_{i,k}(t)) \]
\[ \omega(t) = \sum_{i=1}^{n} (\omega_i)(N_{i,k}(t)) \]

\[ x(t) = x_H(t) / \omega(t) \]
\[ y(t) = y_H(t) / \omega(t) \]
\[ z(t) = z_H(t) / \omega(t) \]
Non-Uniform Rational B-Splines

• A piecewise rational curve is thus defined by:

\[ P(t) = \sum_{i=1}^{n} R_{i,k}(t)P_i, \quad t_{\text{min}} \leq t \leq t_{\text{max}} \]

with supporting rational basis functions:

\[ R_{i,k}(t) = \frac{\omega_i N_{i,k}(t)}{\sum_{j=1}^{n} \omega_j N_{j,k}(t)} \]

This is essentially an average re-weighted by the \( \omega \)'s.

• Such a curve can be made to pass arbitrarily far or near to a control point by changing the corresponding weight.
Non-Uniform Rational B-Splines in action

Weights

Spline

Control functions

Demo
References
