Continuity of the fixpoint operator

Let $D$ be a domain.

By Tarski’s Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\text{fix}(f) \in D$.

**Proposition.** The function

$$\text{fix} : (D \to D) \to D$$

is continuous.

Proof just uses defining properties of \text{fix} — $(\text{fp} 1) \& (\text{fp} 2)$ rather than the explicit construction $\text{fix}(f) = \bigcup_{n \geq 0} f^n(\bot)$. 
Pre-fixed points

Let $D$ be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of $f$, if it exists, will be written $\text{fix}(f)$.

It is thus (uniquely) specified by the two properties:

\begin{align*}
f(\text{fix}(f)) & \sqsubseteq \text{fix}(f) & \text{(lfp1)} \\
\forall d \in D. \ f(d) \sqsubseteq d \Rightarrow \text{fix}(f) & \sqsubseteq d. & \text{(lfp2)}
\end{align*}
\( \text{fix} : (D \to D) \to D \)

is monotone: if \( f \leq f' \) in \( D \to D \), then

\[
\text{fix}(f) \leq \text{fix}(f') \leq \text{fix}(f) 
\]
\( \text{fix} : (D \rightarrow D) \rightarrow D \)

is monotone: if \( f \preceq f' \) in \( D \rightarrow D \), then

\[ f(\text{fix}\, f') \preceq f'(\text{fix}\, f') \preceq \text{fix}\, f' \]

so \( \text{fix}\, f' \) is a pre-fixed point of \( f \)

so by (1fp2) \( \text{fix}\, f \preceq \text{fix}\, f' \)
\[ \text{fix} : (D \to D) \to D \]

is continuous: given \( f_0 \leq f_1 \leq f_2 \leq \ldots \) in \( D \to D \)

want to show \( \text{fix} (\bigcup_{n \geq 0} f_n) \subseteq \bigcup_{n \geq 0} \text{fix}(f_n) \)

By \((\text{fp2})\), enough to show

\( (\bigcup_{n \geq 0} f_n)(d) \leq d \) for \( d = \bigcup_{n \geq 0} \text{fix}(f_n) \)
\text{fix} : (D \to D) \to D

\text{is continuous : given } f_0 \leq f_1 \leq f_2 \leq \ldots \text{ in } D \to D

want to show \( \text{fix} \left( \bigcup_{n \geq 0} f_n \right) \subseteq \bigcup_{n \geq 0} \text{fix}(f_n) \)

By (lfp2), enough to show
\( \left( \bigcup_{n \geq 0} f_n \right)(d) \subseteq d \text{ for } d = \bigcup_{n \geq 0} \text{fix}(f_n) \)

But \( \left( \bigcup_{n \geq 0} f_n \right)(d) = \left( \bigcup_{n \geq 0} f_n \right) \left( \bigcup_{m \geq 0} \text{fix}(f_m) \right) \)

\[ = \bigcup_{n \geq 0} \bigcup_{m \geq 0} f_n \left( \text{fix}(f_m) \right) \]

\[ = \bigcup_{k \geq 0} f_k \left( \text{fix}(f_k) \right) \]

\[ \subseteq \bigcup_{k \geq 0} \text{fix}(f_k) = d \]

(\text{lfp1) for each } f_k
Topic 4

Scott Induction
Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 . d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$. 
Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0. \ d_n \in S) \implies \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called chain-closed (resp. admissible) iff $\{d \in D \mid \Phi(d)\}$ is a chain-closed (resp. admissible) subset of $D$. 

Scott’s Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain $D$.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of $f$ is in $S$, i.e. that

$$\text{fix}(f) \in S,$$

it suffices to prove

$$\forall d \in D \ (d \in S \Rightarrow f(d) \in S).$$
Let $f : D \rightarrow D$ be a continuous function on a domain $D$. Then

- $f$ possesses a least pre-fixed point, given by
  \[ \text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot). \]

- Moreover, $\text{fix}(f)$ is a fixed point of $f$, i.e. satisfies $f(\text{fix}(f)) = \text{fix}(f)$, and hence is the least fixed point of $f$.

where
\[
\begin{align*}
  f^0(\bot) & \triangleq \bot \\
  f^{n+1}(\bot) & \triangleq f(f^n(\bot))
\end{align*}
\]
Proof of the Scott Induction Principle

If we know \( \forall d \in D, d \in S \Rightarrow f(d) \in S \), then \( \bot \in S \) since \( S \) is admissible.
Proof of the Scott Induction Principle

If we know $\forall d \in D. \, d \in S \Rightarrow f(d) \in S$, then

$\bot \in S$

so $f(\bot) \in S$

since $S$ is admissible by
Proof of the Scott Induction Principle

If we know \( \forall d \in D. \ d \in S \Rightarrow f(d) \in S \), then

\[ \bot \in S \quad \text{since } S \text{ is admissible} \]

so \( f(\bot) \in S \)

so \( f(f(\bot)) \in S \) by

\[ f^n(\bot) \in S \quad \text{for all } n \in \mathbb{N} \]
Proof of the Scott Induction Principle

If we know $\forall d \in D, \; d \in S \Rightarrow f(d) \in S$, then

$\bot \in S$ since $S$ is admissible

So $f(\bot) \in S$

So $f(f(\bot)) \in S$

\[ \vdots \]

$f^n(\bot) \in S$ for all $n \in \mathbb{N}$

Hence $\bigcup_{n \geq 0} f^n(\bot) \in S$ since $S$ is admissible

That is, $\text{fix}(f) \in S$
Example 4.2.1

Given \( \{ \text{domain } D \) \)

\( \) continuous function \( f : D \times D \times D \to D \)

then \( g : D \times D \to D \times D \)

is continuous.

\( g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \)

So by Tarski's FPT we get \( \text{fix}(g) \in D \times D \).

Claim: \( u_1 = u_2 \), where \( (u_1, u_2) = \text{fix}(g) \)

Proof: by Scott Induction...
Example 4.2.1

\[
g : D \times D \to D \times D
\]

\[
g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2))
\]

**Claim:** \( u_1 = u_2 \), where \((u_1, u_2) = \text{fix}(g)\)

**Proof**
\[
\Delta \overset{\Delta}{=} \{(d, d) \mid d \in D\}
\]
is an admissible subset of \(D \times D\) because

- \((\perp, \perp) \in \Delta\)
- \((d_0, d'_0) \leq (d_1, d'_1) \leq \ldots \) & \(\forall n. (d_n, d'_n) \in \Delta\) implies
  \[
  \bigcup_{n \geq 0} (d_n, d'_n) = (\bigcup_{n \geq 0} d_n, \bigcup_{n \geq 0} d'_n) = (\bigcup_{n \geq 0} d_n, \bigcup_{n \geq 0} d'_n) \in \Delta
  \]
Example 4.2.1

\[
\begin{aligned}
g : D \times D &\rightarrow D \times D \\
g(d_1, d_2) &= (f(d_1, d_1, d_2), f(d_1, d_2, d_2))
\end{aligned}
\]

Claim: \( u_1 = u_2 \), where \((u_1, u_2) = \text{fix}(g)\)

Proof

\[\Delta = \{(d, d) \mid d \in D\} \text{ admissible}\]

and \( \forall (d, d') \in D \times D. \ (d, d') \in \Delta \Rightarrow g(d, d') \in \Delta \)

because

\[(d, d') \in \Delta \Rightarrow d = d' \]

\[\Rightarrow g(d, d') = (f(d, d, d), f(d, d, d)) \in \Delta \]
Example 4.2.1

\[
\begin{align*}
g : D \times D &\rightarrow D \times D \\
g(d_1, d_2) &= (f(d_1, d_1, d_2), f(d_1, d_2, d_2))
\end{align*}
\]

**Claim:** \( u_1 = u_2 \), where \( (u_1, u_2) = \text{fix}(g) \)

**Proof**

\[\Delta = \{(d, d) \mid d \in D\} \text{ admissible}\]

and \( \forall (d, d') \in D \times D. \ (d, d') \in \Delta \Rightarrow g(d, d') \in \Delta \)

So by Scott Induction

\[\text{fix}(g) \in \Delta\]

Q.E.D.
Example (III): Partial correctness

Let $\mathcal{F} : \text{State} \rightarrow \text{State}$ be the denotation of

\[
\text{while } X > 0 \text{ do } (Y := X \cdot Y; X := X - 1) .
\]

For all $x, y \geq 0$,

\[
\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow \\
\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].
\]
Recall that $F = \text{fix}(f)$ where

$f : (\text{State} \to \text{State}) \to (\text{State} \to \text{State})$

is given by

$$f(w)[x \mapsto x, y \mapsto y] = \begin{cases} [x \mapsto x, y \mapsto y] & \text{if } x \leq 0 \\ w[x \mapsto x-1, y \mapsto xy] & \text{if } x > 0 \end{cases}$$

for all $w \in \text{State} \to \text{State}$ & $x, y \in \mathbb{Z}$
Proof by Scott induction.

We consider the admissible subset of \((\text{State} \rightarrow \text{State})\) given by

\[
S = \left\{ w \mid \begin{array}{l}
\forall x, y \geq 0.
\quad w[X \mapsto x, Y \mapsto y] \downarrow \\
\quad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y]
\end{array} \right\}
\]

and show that

\[
w \in S \implies f(w) \in S.
\]

From now on, let's just write \([x \mapsto x, y \mapsto y]\) as \((x, y)\)
(i.e. identify \text{State} with \(\mathbb{Z} \times \mathbb{Z}\)).
Suppose \( w \in S \). Want to show \( f(w) \in S \), i.e.

\[
x, y \geq 0 \quad \& \quad f(w)(x, y) \downarrow \Rightarrow f(w)(x, y) = (0, !x \cdot y)
\]

So suppose \( x, y \geq 0 \quad \& \quad f(w)(x, y) \downarrow \)

Case \( x = 0 \):

\[
f(w)(x, y) = (x, y) = (0, y) = (0, !0 \cdot y) = (0, !x \cdot y)
\]

by def. of \( f \)

\( x = 0 \)

Since \( 0! = 1 \)

Since \( x = 0 \)
Suppose \( w \in S \). Want to show \( f(w) \in S \), i.e.

\[ x, y \geq 0 \quad \& \quad f(w)(x, y) \downarrow \quad \Rightarrow \quad f(w)(x, y) = (0, ! x \cdot y) \]

So suppose \( x, y \geq 0 \quad \& \quad f(w)(x, y) \downarrow \)

Case \( x > 0 \) : Since \( f(w)(x, y) \downarrow \)

get \( w(x-1, x \cdot y) \downarrow \) by definition of \( f \)
Suppose \( w \in S \). Want to show \( f(w) \in S \), i.e.

\[
x, y \geq 0 \ \& \ f(w)(x, y) \downarrow \Rightarrow f(w)(x, y) = (0, !x \cdot y)
\]

So suppose \( x, y \geq 0 \ \& \ f(w)(x, y) \downarrow \)

Case \( x > 0 \) : Since \( f(w)(x, y) \downarrow \)

get \( w(x-1, x \cdot y) \downarrow \) \ by \ definition \ of \( f \)

But \( x-1, x \cdot y \geq 0 \ \& \ w \in S \)

so \( w(x-1, x \cdot y) = (0, !(x-1) \cdot (x \cdot y)) \) \ by \ def. \ of \( S \)
Suppose \( w \in S \). Want to show \( f(w) \in S \), i.e.

\[
x, y \geq 0 \quad \& \quad f(w)(x, y) \downarrow \implies f(w)(x, y) = (0, !x \cdot y)
\]

So suppose \( x, y \geq 0 \quad \& \quad f(w)(x, y) \downarrow \)

Case \( x > 0 \): Since \( f(w)(x, y) \downarrow \)

get \( w(x-1, x \cdot y) \downarrow \) by definition of \( f \)

But \( x-1, x \cdot y \geq 0 \quad \& \quad w \in S \)

so \( w(x-1, x \cdot y) = (0, !(x-1) \cdot (x \cdot y)) \) by def. of \( S \)

\[
= (0, !x \cdot y)
\]

so \( f(w)(x, y) = w(x-1, x \cdot y) = (0, !x \cdot y) \quad \checkmark
\]

↑ def. of \( f \)
Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset

$$\downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

of $D$ is chain-closed.
Building chain-closed subsets (I)

Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset
  \[
  \downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}
  \]
  of $D$ is chain-closed.

- The subsets
  \[
  \{(x, y) \in D \times D \mid x \sqsubseteq y\}
  \]
  and
  \[
  \{(x, y) \in D \times D \mid x = y\}
  \]
  of $D \times D$ are chain-closed.
Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If $S$ is a chain-closed subset of $E$ then the inverse image

$$f^{-1}S = \{ x \in D \mid f(x) \in S \}$$

is a chain-closed subset of $D$. 
Building chain-closed subsets (II)

Inverse image:

Let $f : D \rightarrow E$ be a continuous function.

If $S$ is a chain-closed subset of $E$ then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is a chain-closed subset of $D$.

Proof: if $d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots$ in $D$ & $\forall n. d_n \in f^{-1}S$
Inverse image:

Let \( f : D \rightarrow E \) be a continuous function.

If \( S \) is a chain-closed subset of \( E \) then the inverse image

\[
f^{-1}S = \{ x \in D \mid f(x) \in S \}
\]

is a chain-closed subset of \( D \).

Proof: if \( d_0 \leq d_1 \leq d_2 \leq \ldots \) in \( D \) & \( \forall n. \ d_n \in f^{-1}S \)
then \( \forall n. \ f(d_n) \in S \), so \( \bigcup_{n \geq 0} f(d_n) \in S \) (\( \text{cos } S \text{ ch.-cl.} \))
**Building chain-closed subsets (II)**

**Inverse image:**

Let \( f : D \to E \) be a continuous function.

If \( S \) is a chain-closed subset of \( E \) then the inverse image

\[
f^{-1}S = \{ x \in D \mid f(x) \in S \}
\]

is a chain-closed subset of \( D \).

**Proof:** If \( d_0 \leq d_1 \leq d_2 \leq \ldots \) in \( D \) and \( \forall n \quad d_n \in f^{-1}S \) then \( \forall n \quad f(d_n) \in S \), so \( \bigcup_{n=0}^{\infty} f(d_n) \in S \) (cos \( S \) ch.-cl.)

So \( f(\bigcup_{n=0}^{\infty} d_n) \in S \) (cos \( f \) cts.)

So \( \bigcup_{n=0}^{\infty} d_n \in f^{-1}S \)
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$

**Proof by Scott induction.**

Consider the admissible property $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \implies g(f(x)) \sqsubseteq g(g(x)) \implies f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)).$$
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$

**Proof by Scott induction.**

Consider the admissible property $\Phi(x) \equiv (f(x) \subseteq g(x))$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \implies g(f(x)) \sqsubseteq g(g(x)) \implies f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)) \sqsubseteq \text{fix}(g) \text{ by (lfp1) for } g$$

so by (lfp2) for $f$, we have

$$\text{fix}(f) \sqsubseteq \text{fix}(g) \quad \Box.$$
Logical operations:

- If \( S, T \subseteq D \) are chain-closed subsets of \( D \) then \( S \cup T \) and \( S \cap T \) are chain-closed subsets of \( D \).
- If \( \{ S_i \}_{i \in I} \) is a family of chain-closed subsets of \( D \) indexed by a set \( I \), then \( \bigcap_{i \in I} S_i \) is a chain-closed subset of \( D \).
- If a property \( P(x, y) \) determines a chain-closed subset of \( D \times E \), then the property \( \forall x \in D. P(x, y) \) determines a chain-closed subset of \( E \).
$S,T$ chain-closed $\Rightarrow S \cup T$ chain-closed

Suppose $d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots$ in $D$ & $\forall n. d_n \in S \cup T$.

If $\bigcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigcup_{n \geq 0} d_n \notin S$.

For each $m \geq 0$, 

$(\forall n \geq m. d_n \in S) \Rightarrow \bigcup_{n \geq 0} d_n = \bigcup_{n \geq m} d_n \in S$.
\( S, T \) chain-closed \( \Rightarrow \) \( S \cup T \) chain-closed

Suppose \( d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots \) in \( D \) \& \( \forall n \in \mathbb{N}, d_n \in S \cup T \)

If \( \bigcup_{n \geq 0} d_n \in S \), we are done.

So suppose \( \bigcup_{n \geq 0} d_n \notin S \)

For each \( m \geq 0 \),

\[
(\forall n \geq m. d_n \in S) \Rightarrow \bigcup_{n \geq 0} d_n = \bigcup_{n \geq m} d_n \in S
\]

So \( \neg (\forall n \geq m. d_n \in S) \)

i.e. \( \exists n \geq m. d_n \in T \) since
$S,T \text{ chain-closed } \Rightarrow S \cup T \text{ chain-closed}$

Suppose $d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots$ in $D$ & $\forall n. d_n \in S \cup T$

If $\bigcup_{n>0} d_n \in S$, we are done.

So suppose $\bigcup_{n>0} d_n \notin S$

For each $m \geq 0$, $\exists n \geq m. d_n \in T$
$S, T$ chain-closed $\Rightarrow SU T$ chain-closed

Suppose $d_0 \leq d_1 \leq d_2 \leq \ldots$ in $D$ & $\forall n. d_n \in SU T$

If $\bigcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$, $\exists n \geq m. d_n \in T$

So we can choose $n_0 \leq n_1 \leq n_2 \leq \ldots$ satisfying $\forall m. m \leq n_m$ & $d_{n_m} \in T$.

So $\bigcup_{n \geq 0} d_n = \bigcup_{m \geq 0} d_{n_m} \in T$  Q.E.D.
Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of $D$ then $S \cup T$ and $S \cap T$ are chain-closed subsets of $D$.

- If $\{S_i\}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D. P(x, y)$ determines a chain-closed subset of $E$.

\[ \text{N.B. in general } \bigcup_{i \in I} S_i = \{ d \mid \exists i. d \in S_i \} \quad \text{and} \quad D - S \text{ need not be chain-closed.} \]
for each $i \in \mathbb{N}$

$S_i = \{0, 1, 2, \ldots, i\}$ is chain-closed subset
of the domain $\Omega = \left\{ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right\}$

but

$\bigcup_{i \in \mathbb{N}} S_i = \mathbb{N}$ is not a chain-closed subset
of the domain $\Omega = \left\{ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right\}$
\[ S = \{0,2,4,\ldots\} \cup \{\omega\} \text{ is chain-closed subset of the domain } \Omega = \left\{ \begin{array}{c} \vdots \\ \omega \\ \vdots \\ \downarrow \\ \omega_0 \end{array} \right\} \]

but
\[ D - S = \{1,3,5,\ldots\} \]

is \underline{not} a chain-closed subset of the domain \[ \Omega = \left\{ \begin{array}{c} \vdots \\ \omega \\ \vdots \\ \downarrow \\ \omega_0 \end{array} \right\} \]