A chain complete poset, or cpo for short, is a poset \((D, \sqsubseteq)\) in which all countable increasing chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) have least upper bounds, \(\bigsqcup_{n \geq 0} d_n\):

\[
\forall m \geq 0. \ d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \quad \text{(lub1)}
\]

\[
\forall d \in D. (\forall m \geq 0. \ d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \quad \text{(lub2)}
\]

A domain is a cpo that possesses a least element, \(\bot\):

\[
\forall d \in D. \bot \sqsubseteq d.
\]
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that

$$f \left( \bigsqcup_{n \geq 0} d_n \right) = \bigsqcup_{n \geq 0} f(d_n) \text{ in } E.$$
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that
    
    $$f\left(\bigsqcup_{n \geq 0} d_n\right) = \bigsqcup_{n \geq 0} f\left(d_n\right) \text{ in } E.$$ 

    NB
    
    $$f(d_0) \subseteq f(d_1) \subseteq f(d_2) \subseteq \ldots$$

    $\Rightarrow$ $f$ monotone

    $$\forall i. d_i \sqsubseteq \bigcup_{n \geq 0} d_n \Rightarrow \forall i. f(d_i) \sqsubseteq f\left(\bigcup_{n \geq 0} d_n\right)$$

    $$\Rightarrow \bigcup_{n \geq 0} f\left(d_i\right) \sqsubseteq f\left(\bigcup_{n \geq 0} d_n\right)$$
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that

  \[ f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E. \]

  \[ \text{NB} \quad \forall i: d_i \sqsubseteq \bigcup_{n \geq 0} d_n \quad \text{monotonicity} \quad \Rightarrow \quad \forall i: f(d_i) \sqsubseteq f\left( \bigcup_{n \geq 0} d_n \right) \]

  \[ \Rightarrow \quad \bigcup_{i \geq 0} f(d_i) \sqsubseteq f\left( \bigcup_{n \geq 0} d_n \right) \]

  So given 1, for 2 just need $f(\bigcup_{n \geq 0} d_n) \sqsubseteq \bigcup_{n \geq 0} f(d_n)$
Continuity and strictness

• If $D$ and $E$ are cpo’s, the function $f$ is continuous iff

1. it is monotone, and

2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that

$$f\left(\bigsqcup_{n \geq 0} d_n\right) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$ 

• If $D$ and $E$ have least elements, then the function $f$ is strict iff $f(\bot) = \bot$. 

38
Tarski’s Fixed Point Theorem

Let \( f : D \to D \) be a continuous function on a domain \( D \). Then

- \( f \) possesses a least pre-fixed point, given by
  \[
  \text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).
  \]

- Moreover, \( \text{fix}(f) \) is a fixed point of \( f \), i.e. satisfies
  \[
  f(\text{fix}(f)) = \text{fix}(f),
  \]
  and hence is the least fixed point of \( f \).

where
\[
\begin{cases}
  f^0(\bot) \triangleq \bot \\
  f^{n+1}(\bot) \triangleq f(f^n(\bot))
\end{cases}
\]
Let $D$ be a poset and $f : D \rightarrow D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of $f$, if it exists, will be written $\text{fix}(f)$.

It is thus (uniquely) specified by the two properties:

$$f(\text{fix}(f)) \sqsubseteq \text{fix}(f) \quad \text{(lfp1)}$$

$$\forall d \in D. \ f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d. \quad \text{(lfp2)}$$
Proof of Tarski’s Theorem

\[ \bot \in f(\bot) \]  because \( \bot \) is least elt of D
Proof of Tarski's Theorem

\[ \bot \subseteq f(\bot) \quad \text{because } \bot \text{ is least elt of } D \]

so \[ f(\bot) \subseteq f(f(\bot)) \subseteq f^2(\bot) \quad \text{by monotonicity of } f \]

so \[ f^2(\bot) \subseteq f(f^2(\bot)) = f^3(\bot) \]

etc.
Proof of Tarski's Theorem

\[ \bot \subseteq f(\bot) \text{ because } \bot \text{ is least elt of } D \]

so \[ f(\bot) \subseteq f(f(\bot)) \subseteq f^2(\bot) \text{ by monotonicity of } f \]

so \[ f^2(\bot) \subseteq f(f^2(\bot)) = f^3(\bot) \]

etc.

We get a chain \[ \bot \subseteq f(\bot) \subseteq f^2(\bot) \subseteq f^3(\bot) \subseteq \ldots \]
and can form its lub \[ \bigcup_{n \geq 0} f^n(\bot) \]
Proof of Tarski’s Theorem

Applying $f$ to $\bot \leq f(\bot) \leq f^2(\bot) \leq \cdots \leq \bigcup_{n \geq 0} f^n(\bot)$ we get

$f(\bot) \leq f(f(\bot)) \leq f(f^2(\bot)) \leq \cdots \leq f\left( \bigcup_{n \geq 0} f^n(\bot) \right)$

by monotonicity of $f$
Proof of Tarski’s Theorem

Applying $f$ to

$$
\bot \subseteq f(\bot) \subseteq f^2(\bot) \subseteq \cdots \subseteq \bigcup_{n \geq 0} f^n(\bot)
$$

we get

$$
f(\bot) \subseteq f(f(\bot)) \subseteq f(f^2(\bot)) \subseteq \cdots \subseteq f\left( \bigcup_{n \geq 0} f^n(\bot) \right)
$$

by continuity of $f$

$$
\bigcup_{n \geq 0} f(f^n(\bot))
$$

$$
\bigcup_{n \geq 0} f^{n+1}(\bot)
$$

$$
\bigcup_{m \geq 1} f^m(\bot)
$$
Proof of Tarski's Theorem

So $\bigcup_{n \geq 0} f^n(\perp)$ is a (pre-)fixed point for $f$

\[
\bigcup_{n \geq 0} f^n(\perp) = \bigcup_{m \geq 1} f^m(\perp)
\]
Proof of Tarski’s Theorem

For any pre-fixed point \( f(d) \leq d \) we have \( \bot \leq d \) because \( \bot \) is least elt of \( D \).
Proof of Tarski’s Theorem

For any pre-fixed point \( f(d) \equiv d \) we have \( \bot \subseteq d \) because \( \bot \) is least elt of \( D \).

So \( f(\bot) \subseteq f(d) \subseteq d \) by monotonicity.
Proof of Tarski’s Theorem

For any pre-fixed point \( f(d) \subseteq d \) we have

\[ \bot \subseteq d \quad \text{because } \bot \text{ is least elt of } D \]

So \( f(\bot) \subseteq f(d) \subseteq d \) \hspace{1cm} \text{monotonicity +}

So \( f^2(\bot) = f(f(\bot)) \subseteq f(d) \subseteq d \)

etc.
Proof of Tarski’s Theorem

For any pre-fixed point \( f(d) \leq d \) we have
\[ \bot \leq d \] because \( \bot \) is least elt of \( D \)

So \( f(\bot) \leq f(d) \leq d \)

So \( f^2(\bot) = f(f(\bot)) \leq f(d) \leq d \)

etc.

We get \( f^n(\bot) \leq d \) for all \( n \geq 0 \)

So \( \bigcup_{n \geq 0} f^n(\bot) \leq d \)
Proof of Tarski’s Theorem

For any pre-fixed point \( f(d) \equiv d \) we have

We get

\[
\bigcup_{n \geq 0} f^n(\bot) \subseteq d
\]

So \( \bigcup_{n \geq 0} f^n(\bot) \) is a least pre-fixed point.

QED
Fixed point property of
\[[\text{while } B \text{ do } C]\]

\[[\text{while } B \text{ do } C] = f_{[B],[C]}([\text{while } B \text{ do } C])

where, for each \(b : \text{State} \rightarrow \{\text{true, false}\}\) and \(c : \text{State} \rightarrow \text{State}\), we define

\(f_{b,c} : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})\)

as

\[f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}) . \lambda s \in \text{State}. \text{if } (b(s), w(c(s)), s).\]

- Why does \(w = f_{[B],[C]}(w)\) have a solution?
- What if it has several solutions—which one do we take to be \([\text{while } B \text{ do } C]\)?
Fixed point property of
\[ \text{[while } B \text{ do } C] \]

\[ \text{[while } B \text{ do } C] = f_{[B],[C]}(\text{[while } B \text{ do } C]) \]

where, for each \( b : \text{State} \rightarrow \{\text{true, false}\} \) and \( c : \text{State} \rightarrow \text{State} \), we define

\[ f_{b,c} : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State}) \]

as

\[ f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \]

\[ \text{if } (b(s), w(c(s)), s) . \]

- Why does \( w = f_{[B],[C]}(w) \) have a solution?
- What if it has several solutions—-which one do we take to be \( \text{[while } B \text{ do } C] \)?

\( \text{least (pre-)fixed point} \)
Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c \subseteq c_2 \subseteq \ldots$ in State $\rightarrow$ State

$$f_{b,c}(U_{n \geq 0} c_n) = \lambda s \in \text{State. if } (b(s), (U_{n \geq 0} c_n)(c(s)), s)$$

That is

$$f_{b,c}(U_{n \geq 0} c_n) = \left\{ (s, s') \mid \begin{array}{l}
    b(s) = \text{true} \land \exists s''. (c(s) = s'' \land (U_{n \geq 0} c_n)(s'') = s') \\
    b(s) = \text{false} \land s = s'
\end{array} \right\}$$
Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c_1 \subseteq c_2 \subseteq \ldots$ in State → State

$$f_{b,c} \left( \bigcup_{n \geq 0} c_n \right) = \lambda s \in \text{State}. \text{if } (b(s), (\bigcup_{n \geq 0} c_n)(c(s)), s)$$

that is

$$f_{b,c} \left( \bigcup_{n \geq 0} c_n \right) = \left\{ (s, s') \mid b(s) = \text{true} \land \exists s''. c(s) = s'' \land \exists n \geq 0. c_n(s'') = s' \lor b(s) = \text{false} \land s = s' \right\}$$
Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c \subseteq c_2 \subseteq \ldots$ in State $\rightarrow$ State

$$f_{b,c} \left( U_{n \geq 0} \mathcal{C}_n \right) = \lambda s \in \text{State}. \text{ if } (b(s), \left( U_{n \geq 0} \mathcal{C}_n \right) (c(s)), s)$$

That is

$$f_{b,c} \left( U_{n \geq 0} \mathcal{C}_n \right) = \left\{ (s, s') \left| \begin{array}{l}
\exists n \geq 0. b(s) = \text{true} \land \exists s''. c(s) = s'' \land c_n (s'') = s' \land \right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
Continuity of $f_{b,c}$

Suppose $c_0 \leq c \leq c_2 \leq \ldots$ in State $\Rightarrow$ State

$$f_{b,c} \left( \bigcup_{n \geq 0} c_n \right) = \lambda s \in \text{State}. \text{ if } (b(s), (\bigcup_{n \geq 0} c_n)(c(s)), s)$$

That is

$$f_{b,c} \left( \bigcup_{n \geq 0} c_n \right) = \left\{ (s,s') \left| \begin{array}{l}
\exists n \geq 0. b(s) = \text{true} \land \exists s''. (c(s) = s'' \land c_n(s'') = s') \\
\lor b(s) = \text{false} \land s = s'
\end{array} \right. \right\}$$

$$= \bigcup_{n \geq 0} \left\{ (s,s') \left| \text{if} (b(s), c_n(c(s)), s) = s' \right. \right\}$$

$$= \bigcup_{n \geq 0} f_{b,c} (c_n)$$

QED
\[ \textbf{[while } B \text{ do } C \text{]} \]

\[ \textbf{[while } B \text{ do } C \text{]} \]

\[ = \text{fix}(f_{B}, f_{C}) \]

\[ = \bigcup_{n \geq 0} f_{B}, f_{C}^{n}(\bot) \]

\[ = \lambda s \in \text{State.} \]

\[ \begin{cases} 
[C]^{k}(s) & \text{if } k \geq 0 \text{ is such that } [B](C)^{k}(s)) = \text{false} \\
\text{undefined} & \text{if } [B](C)^{i}(s)) = \text{true} \text{ for all } i \geq 0 \end{cases} \]

\[ \text{requires proof...} \]
Example

Domain \( D = (\mathcal{P}(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \rightarrow \uparrow \))

Function \( f : D \rightarrow D \)

\[
f(S) = \{0\} \cup \{x+2 \mid x \in S\}
\]
Example

Domain \( D = (\mathcal{P}(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \rightarrow \mathbb{N} \))

Function \( f : D \rightarrow D \)

\[
f(S) = \{0\} \cup \{x+2 \mid x \in S\}
\]

\( S \in D \) is a prefixed point of \( f \) if \( f(S) \subseteq S \)

i.e. \( 0 \in S \) \& \( x+2 \in S \) for all \( x \in S \)

i.e. \( S \) is closed under the rules \( 0 \in S \) \& \( x \in S \) \( \Rightarrow x+2 \in S \)
Example

Domain \( D = (P(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \to \mathbb{N} \))

Function \( f : D \to D \)

\[
f(S) = \{ 0 \} \cup \{ x+2 \mid x \in S \}
\]

\( S \in D \) is a pre-fixed point of \( f \) if

\[
f(S) \subseteq S
\]

i.e. \( 0 \in S \) & \( x+2 \in S \) for all \( x \in S \)

i.e. \( S \) is closed under the rules

\[
\begin{align*}
0 & \in S \\
x + 2 & \in S 
\end{align*}
\]

So expect least pre-fixed point of \( f \) to be \( \text{Even} = \{ 2x \mid x \in \mathbb{N} \} \)
Example

Domain \( D = (P(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \rightarrow \mathbb{1} \))

Function \( f : D \rightarrow D \)

\[
f(S) = \{0\} \cup \{x + 2 | x \in S\}
\]

\( f \) is monotone: \( S \subseteq S' \Rightarrow f(S) \subseteq f(S') \) \( \checkmark \)
Example

Domain \( D = (\mathcal{P}(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \to 1 \))

Function \( f : D \to D \)

\[
f(S) \triangleq \{0\} \cup \{x+2 \mid x \in S\}
\]

\( f \) is monotone: \( S \subseteq S' \Rightarrow f(S) \subseteq f(S') \) \( \checkmark \)

\( f \) is continuous:

\[
f(\bigcup_{n \geq 0} S_n) = \{0\} \cup \{x+2 \mid x \in \bigcup_{n \geq 0} S_n\}
\]

\[
= \{0\} \cup \bigcup_{n \geq 0} \{x+2 \mid x \in S_n\}
\]

\[
= \bigcup_{n \geq 0} f(S_n) \quad \checkmark
\]
Example

Domain $D = (\mathcal{P}(\mathbb{N}), \subseteq)$ (same as $\mathbb{N} \to \mathbb{N}$)

Function $f : D \to D$

$f(S) = \{0\} \cup \{x+2 \mid x \in S\}$

Tarski's Theorem applies:

$\text{fix}(f) = \bigcup_{n \geq 0} f^n(\emptyset)$

$f(\emptyset) = \{0\}$

$f^2(\emptyset) = \{0\} \cup \{0+2\}$

$f^3(\emptyset) = \{0, 2, 4\}$

$f^n(\emptyset) = \{0, 2, 4, \ldots, 2(n-1)\}$
Example

Domain \( D = (\mathcal{P}(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \to \mathbb{N} \))

Function \( f : D \to D \)

\[
f(S) \triangleq \{0\} \cup \{x+2 | x \in S\}
\]

Tarski's Theorem applies:

\[
\text{fix}(f) = \bigcup_{n \geq 0} f^n(\emptyset) = \{0, 2, 4, 8, \ldots\}
\]

\[
= \{2^x | x \in \mathbb{N}\}
\]

(as expected).

\[
\begin{align*}
 f(\emptyset) &= \{0\} \\
f^2(\emptyset) &= \{0\} \cup \{0+2\} \\
f^3(\emptyset) &= \{0, 2, 4\} \\
 f^n(\emptyset) &= \{0, 2, 4, \ldots, 2^{(n-1)}\}
\end{align*}
\]