Lectures 1–3

Exercise 1  Let $\Omega$ be the domain of “vertical natural numbers” pictured in Figure 1 of the lecture notes.

(i) Is every monotone function from $\Omega$ to $\Omega$ continuous?

(ii) Does every monotone function from $\Omega$ to $\Omega$ have a least prefixed point?

Exercise 2  For a partially ordered set $(P, \sqsubseteq)$, let $(\text{Ch}(P), \sqsubseteq_{\text{ptw}})$ be the partially ordered set of chains in $P$ ordered pointwise. That is $\text{Ch}(P) \overset{\text{def}}{=} \{x = \{x_n\}_{n \in \mathbb{N}} \mid \text{for all } i \leq j \text{ in } \mathbb{N}, x_i \sqsubseteq x_j \text{ in } P\}$ and $x \sqsubseteq_{\text{ptw}} x' \overset{\text{def}}{=} x_n \sqsubseteq x'_n \text{ for all } n \in \mathbb{N}$

Show that if $P$ is a domain then so is $\text{Ch}(P)$.

Exercise 3  

(i) Let $D_1, D_2$ and $E$ be domains. Show that a function $f : D_1 \times D_2 \rightarrow E$ is continuous if it is continuous in each argument separately, that is, if for all $d_1 \in D_1$ and $d_2 \in D_2$, the functions $f(d_1, -) : D_2 \rightarrow E$ and $f(-, d_2) : D_1 \rightarrow E$ are continuous.

(ii) Let $\mathcal{O}$ be the domain with two elements $\bot \sqsubseteq \top$. For a domain $E$ and $e \in E$, define the function $g_e : E \rightarrow \mathcal{O}$ by

$$g_e(x) = \begin{cases} 
\bot & \text{if } x \sqsubseteq e \\
\top & \text{if } x \not\sqsubseteq e 
\end{cases}$$

Show that $g_e$ is continuous.

(iii) As an example of the definition in part (ii), let $E = \mathcal{B}_\bot \times \mathcal{B}_\bot$, where $\mathcal{B} = \{\text{true, false}\}$, and consider $g_{\text{false,false}} : E \rightarrow \mathcal{O}$. Show that $g_{\text{false,false}}(x, y) = \top$ if $x = \text{true}$ or $y = \text{true}$.

(iv) Let $f : D \rightarrow E$ be a function between domains $D$ and $E$. Show that $f$ is continuous iff $\forall e \in E. g_e \circ f$ is continuous.

Exercise 4  Let $\mathcal{O}$ be the domain in Exercise 3(ii).

(i) Draw a diagram which represents the elements of the function domain $\mathcal{O} \rightarrow \mathcal{O}$ and shows their ordering.

(ii) Any set $X$ can be considered as a flat domain $X_\bot$ by adding a bottom element. Show that the strict continuous functions $X_\bot \rightarrow \mathcal{O}$ are in 1-1 correspondence with the subsets of $X$. 
Lectures 4–6

Exercise 5  Let $D$ be a domain and $k : D \to D$ a continuous function. Let $\mathbb{B} = \{\text{true}, \text{false}\}$. Define the conditional function $\text{if} : \mathbb{B}_\bot \times D \times D \to D$ by

$$
\text{if}(b, d, d') = \begin{cases} 
  d & \text{if } b = \text{true} \\
  d' & \text{if } b = \text{false} \\
  \bot & \text{if } b = \bot 
\end{cases}
$$

Let $h : D \to \mathbb{B}_\bot$ be a continuous function which is strict (so $h(\bot) = \bot$). The function $f^*$ is the least continuous function from $D$ to $D$ such that

$$
\forall x \in D. \; f^*(x) = \text{if}(h(x), x, f^*(k(x)))
$$

(i) Show that $\forall x \in D. \; h(f^*(x)) = \text{if}(h(x), h(x), h(f^*(k(x))))$.

(ii) Prove that the property $Q(f) \overset{\text{def}}{=} \forall x \in D. \; h(f(x)) \sqsubseteq \text{true}$ is admissible.

(iii) Prove $Q(f^*)$ by fixed point induction.

Exercise 6  Suppose that $D$ is a domain and $f : D \times D \to D$ is a continuous function satisfying the property $\forall d, e \in D. \; f(d, e) = f(e, d)$. Let $g : D \times D \to D \times D$ be defined by

$$
g(d_1, d_2) = (f(d_1, f(d_1, d_2)), f(f(d_1, d_2), d_2))
$$

Let $(u_1, u_2) = \text{fix}(g)$. Show that $u_1 = u_2$ using Scott induction.

Exercise 7  Let $D$ and $E$ be domains and let $f : D \to D$ and $g : E \to E$ be continuous functions.

(i) Define $f \times g : D \times E \to D \times E$ to be the continuous function given by $(f \times g)(d, e) = (f(d), g(e))$ and let $\pi_1 : D \times E \to D$ and $\pi_2 : D \times E \to E$ respectively denote the first and second projection functions. Show that $\text{fix}(f \times g) \sqsubseteq (\text{fix}(f), \text{fix}(g))$ and that $\text{fix}(f) \sqsubseteq \pi_1(\text{fix}(f \times g))$ and $\text{fix}(g) \sqsubseteq \pi_2(\text{fix}(f \times g))$.

(ii) It follows from part (i) that $\text{fix}(f \times g) = (\text{fix}(f), \text{fix}(g))$. Use this and Scott’s Fixed Point Induction Principle to show that, for all strict continuous functions $h : D \to E$,

$$
h \circ f = g \circ h \Rightarrow h(\text{fix}(f)) = \text{fix}(g)
$$

Lectures 7–10

Exercise 8  Which of the following statements are true or false, and why?

(i) For all PCF-types $\tau$ and closed PCF-terms $M_1$ and $M_2$ of type $\tau$, if $[M_1] = [M_2]$ in $[\tau]$ then $M_1 \equiv_{\text{ctx}} M_2 : \tau$.

(ii) For all PCF-types $\tau$ and closed PCF-terms $M_1$ and $M_2$ of type $\tau$, if $M_1 \equiv_{\text{ctx}} M_2 : \tau$, then $[M_1] = [M_2]$ in $[\tau]$.

(iii) For all closed PCF-terms $M_1$ and $M_2$ of type $\text{nat} \to \text{nat}$, if $M_1 \equiv_{\text{ctx}} M_2 : \text{nat} \to \text{nat}$, then $[M_1] = [M_2]$ in $\mathbb{N}_\bot \to \mathbb{N}_\bot$. 
Exercise 9  Consider the following two statements for PCF terms $M_1$ and $M_2$ for which the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold for some type environment $\Gamma$ and type $\tau$.

1) For all PCF contexts $C[-]$ for which $C[M_1] : bool$ and $C[M_2] : bool$,
   $\downarrow_{\text{bool}}$ stands for the existence of a value $V : \tau$ for which $M \downarrow_{\tau} V$.

2) For all PCF contexts $C[-]$ for which $C[M_1] : bool$ and $C[M_2] : bool$,
   $C[M_1] \downarrow_{\text{bool true}} \iff C[M_2] \downarrow_{\text{bool true}}$

(i) Show that (1) implies (2).

(ii) Show that (2) implies that $M_1$ and $M_2$ are contextually equivalent.

Extra

Exercise 10  Suppose that $D$ is a domain and that $\text{lam} : (D \rightarrow D) \rightarrow D$ and $\text{app} : D \rightarrow (D \rightarrow D)$ are continuous functions. Use this data to give a denotational semantics for the terms of the untyped $\lambda$-calculus, by answering question 5 from paper 7 of the 1998 CS Tripos.