CST 2018/19 Part II
Denotational Semantics
Exercise Sheet

Lectures 1–3

Exercise 1 Let Ω de the domain of "vertical natural numbers" pictured in Figure 1 of the lecture notes.

(i) Is every monotone function from Ω to Ω continuous?

(ii) Does every monotone function from Ω to Ω have a least prefixed point?

Exercise 2 For a partially ordered set $(P, \sqsubseteq)$, let $(\mathrm{Ch}(P), \sqsubseteq_{\mathrm{ptw}})$ be the partially ordered set of chains in P ordered pointwise. That is

$$\mathrm{Ch}(P) \stackrel{\text{def}}{=} \{x = \{x_n\}_{n\in\mathbb{N}} \mid \text{for all } i \leq j \text{ in } \mathbb{N},\, x_i \sqsubseteq x_j \text{ in } P\}$$

and

$$x \sqsubseteq_{\mathrm{ptw}} x' \stackrel{\text{def}}{\Leftrightarrow} x_n \sqsubseteq x'_n \text{ for all } n \in \mathbb{N}$$

Show that if P is a domain then so is $\mathrm{Ch}(P)$.

Exercise 3

(i) Let D_1, D_2 and E be domains. Show that a function $f : D_1 \times D_2 \to E$ is continuous if it is continuous is each argument separately, that is, if for all $d_1 \in D_1$ and $d_2 \in D_2$, the functions $f(d_1, -) : D_2 \to E$ and $f(-, d_2) : D_1 \to E$ are continuous.

(ii) Let $\mathbb{O}$ be the domain with two elements $\bot \sqsubseteq \top$. For a domain E and $e \in E$, define the function $g_e : E \to \mathbb{O}$ by

$$g_e(x) = \begin{cases} \bot & \text{if } x \sqsubseteq e \\ \top & \text{if } x \not\sqsubseteq e \end{cases}$$

Show that g_e is continuous.

(iii) As an example of the definition in part (ii), let $E = \mathbb{B}_\bot \times \mathbb{B}_\bot$, where $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$, and consider $g_{(\mathit{false},\mathit{false})} : E \to \mathbb{O}$. Show that $g_{(\mathit{false},\mathit{false})}(x, y) = \top$ iff $x = \mathit{true}$ or $y = \mathit{true}$.

(iv) Let $f : D \to E$ be a function between domains D and E. Show that f is continuous iff $\forall e \in E.\ g_e \circ f$ is continuous.

Exercise 4 Let $\mathbb{O}$ be the domain in Exercise 3(ii).

(i) Draw a diagram which represents the elements of the function domain $\mathbb{O} \to \mathbb{O}$ and shows their ordering.

(ii) Any set X can be considered as a flat domain $X_\bot$ by adding a bottom element. Show that the strict continuous functions $X_\bot \to \mathbb{O}$ are in 1-1 correspondence with the subsets of X.

Lectures 4–6

Exercise 5 Let D be a domain and $k : D \to D$ a continuous function. Let $\mathbb{B} = \{true, false\}$. Define the conditional function $if : \mathbb{B}_\perp \times D \times D \to D$ by

$$if(b, d, d') = \begin{cases} d & \text{if } b = true \\ d' & \text{if } b = false \\ \perp & \text{if } b = \perp \end{cases}$$

Let $h : D \to \mathbb{B}_\perp$ be a continuous function which is strict (so $h(\perp) = \perp$). The function f^* is the least continuous function from D to D such that

$$\forall x \in D.\ f^*(x) = if(h(x), x, f^*(k(x)))$$

(i) Show that $\forall x \in D.\ h(f^*(x)) = if(h(x), h(x), h(f^*(k(x))))$.

(ii) Prove that the property $Q(f) \overset{\text{def}}{\Leftrightarrow} \forall x \in D.\ h(f(x)) \sqsubseteq true$ is admissible.

(iii) Prove $Q(f^*)$ by fixed point induction.

Exercise 6 Suppose that D is a domain and $f : D \times D \to D$ is a continuous function satisfying the property $\forall d, e \in D.\ f(d, e) = f(e, d)$. Let $g : D \times D \to D \times D$ be defined by

$$g(d_1, d_2) = (f(d_1, f(d_1, d_2)), f(f(d_1, d_2), d_2))$$

Let $(u_1, u_2) = fix(g)$. Show that $u_1 = u_2$ using Scott induction.

Exercise 7 Let D and E be domains and let $f : D \to D$ and $g : E \to E$ be continuous functions.

(i) Define $f \times g : D \times E \to D \times E$ to be the continuous function given by $(f \times g)(d, e) = (f(d), g(e))$ and let $\pi_1 : D \times E \to D$ and $\pi_2 : D \times E \to E$ respectively denote the first and second projection functions. Show that $fix(f \times g) \sqsubseteq (fix(f), fix(g))$ and that $fix(f) \sqsubseteq \pi_1(fix(f \times g))$ and $fix(g) \sqsubseteq \pi_2(fix(f \times g))$.

(ii) It follows from part (i) that $fix(f \times g) = (fix(f), fix(g))$. Use this and Scott's Fixed Point Induction Principle to show that, for all strict continuous functions $h : D \to E$,

$$h \circ f = g \circ h \ \Rightarrow\ h(fix(f)) = fix(g)$$

Lectures 7–10

Exercise 8 Which of the following statements are true or false, and why?

(i) For all PCF-types τ and closed PCF-terms M_1 and M_2 of type τ, if $[\![M_1]\!] = [\![M_2]\!]$ in $[\![\tau]\!]$ then $M_1 \cong_{\text{ctx}} M_2 : \tau$.

(ii) For all PCF-types τ and closed PCF-terms M_1 and M_2 of type τ, if $M_1 \cong_{\text{ctx}} M_2 : \tau$, then $[\![M_1]\!] = [\![M_2]\!]$ in $[\![\tau]\!]$.

(iii) For all closed PCF-terms M_1 and M_2 of type $nat \to nat$, if $M_1 \cong_{\text{ctx}} M_2 : nat \to nat$, then $[\![M_1]\!] = [\![M_2]\!]$ in $\mathbb{N}_\perp \to \mathbb{N}_\perp$.

Exercise 9 Consider the following two statements for PCF terms M_1 and M_2 for which the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold for some type environment Γ and type τ.

(1) For all PCF contexts $\mathcal{C}[-]$ for which $\mathcal{C}[M_1] : bool$ and $\mathcal{C}[M_2] : bool$,

$$\mathcal{C}[M_1] \Downarrow_{bool} \;\Leftrightarrow\; \mathcal{C}[M_2] \Downarrow_{bool}$$

where, for a closed term M of type τ, the notation $M \Downarrow_\tau$ stands for the existence of a value $V : \tau$ for which $M \Downarrow_\tau V$.

(2) For all PCF contexts $\mathcal{C}[-]$ for which $\mathcal{C}[M_1] : bool$ and $\mathcal{C}[M_2] : bool$,

$$\mathcal{C}[M_1] \Downarrow_{bool} \mathbf{true} \;\Leftrightarrow\; \mathcal{C}[M_2] \Downarrow_{bool} \mathbf{true}$$

(i) Show that (1) implies (2).

(ii) Show that (2) implies that M_1 and M_2 are contextually equivalent.

Extra

Exercise 10 Suppose that D is a domain and that $lam : (D \to D) \to D$ and $app : D \to (D \to D)$ are continuous functions. Use this data to give a denotational semantics for the terms of the untyped λ-calculus, by answering question 5 from paper 7 of the 1998 CS Tripos.