This example sheet covers material up to Lecture 4 on 15 October. The more challenging questions are 3(b), 5(b), 8, 10(b).

**Question 1.** If $X_1, \ldots, X_n$ are independent samples from Uniform[0, $\theta$], find the maximum likelihood estimator for $\theta$. Hint. Write the density as

$$P_{X}(x \mid \theta) = \frac{1}{\theta} 1_{x \geq 0} 1_{x \leq \theta} \quad \text{for } x \in \mathbb{R}$$

where $1_{\cdot}$ stands for the indicator function, $1_{\text{true}} = 1$ and $1_{\text{false}} = 0$.

**Question 2.** Let $x_i$ be the population of city $i$, and let $y_i$ be the number of crimes reported. Fit the model $Y_i \sim \text{Poisson}(\lambda x_i)$, where $\lambda$ is an unknown parameter.

**Question 3.** A 0/1 signal is being sent over a noisy wire. If $x_n$ is the true signal at timestep $n \in \{1, 2, \ldots\}$, then the received message is $R_n = x_n + \text{Normal}(0, \varepsilon^2)$, where $\varepsilon$ is known. Suppose that the signal being sent has a single changepoint, i.e.

$$x_n = \begin{cases} 
0 & \text{for } n \leq \theta \\
1 & \text{for } n > \theta 
\end{cases}$$

(a) Give pseudocode for a function `changepoint([r_1, \ldots, r_N])` to estimate $\theta$ from $N$ received messages. We’d like to run this as a ‘streaming’ procedure, looking for a changepoint in $[r_1]$ then in $[r_1, r_2]$ then in $[r_1, r_2, r_3]$ and so on.

(b) The naive `changepoint` function might produce lots of false alarms, detecting a changepoint as soon as it sees a high $r_n$ and then realising its mistake when it sees $r_{n+1}$. Suggest an evidence-based approach to fix this problem.

**Question 4.**

(a) Let $U$ be a uniform random variable on [0, 1]. Let $Y = U(1 - U)$. Calculate $P(Y \leq y)$, and hence find the density of $Y$.

(b) A point lightsource at coordinates (0, 1) sends out a ray of light at an angle $\Theta$ chosen uniformly in $[-\pi/2, \pi/2]$. Let $X$ be the point where the ray intersects the horizontal line through the origin. What is the density of $X$? This random variable is known as the Cauchy distribution. It is unusual in that it has no mean.

**Question 5.** The Gumbel distribution is used in econometrics, for modelling how people make choices. If $X \sim \text{Gumbel}(\lambda)$ then

$$P(X \leq x) = \exp[-\exp(-(x - \lambda))], \quad x \in \mathbb{R}.$$ 

Let $X_1 \sim \text{Gumbel}(\lambda_1)$ and $X_2 \sim \text{Gumbel}(\lambda_2)$ be independent. Show the following:

$$\max(X_1, X_2) \sim \text{Gumbel}(\log(e^{\lambda_1} + e^{\lambda_2})) \quad \text{ (a)}$$

and

$$P(X_1 \geq X_2) = \frac{e^{\lambda_1}}{e^{\lambda_1} + e^{\lambda_2}}. \quad \text{ (b)}$$

Hint. For the first equation, $P(\max(X_1, X_2) \leq x) = P(X_1 \leq x \text{ and } X_2 \leq x)$. This trick saves you a fiddly integration.
Question 6. Let $X \sim \text{Normal}(\mu, \sigma^2)$. We wish to sample from $(X \mid X \geq 0)$. Give pseudocode, based on the inversion method. You should use the scipy functions `ppf` and `cdf` functions, described in the lecture notes appendix.

Question 7. Consider a pair of random variables with joint density

$$Pr_{X,Y}(x,y) = \frac{3}{16} xy^2, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2.$$ 

Find $Pr_X(x)$ and $Pr_Y(y)$, the marginal densities. **Hint. It may be easier to first prove Exercise 1.7 from lecture notes.**

Question 8. In a hash table with $n$ buckets and a load factor of $\alpha$ (i.e. with $an$ items hashed), what is the expected number of empty buckets? **Hint: $E_A = P(A)$.**

Question 9. Let $X_i \sim \text{Uniform}(2x/3, 4x/3)$, where $x$ is given. Find a 95% confidence interval for $X_1 + \cdots + X_n$. This arises in the context of statistical multiplexing of TCP flows on the Internet. See Exercise 2.6 in lecture notes for the background.

Question 10. Let $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$, where the $X_i$ are independent $\text{Exp}(\lambda)$ random variables.

(a) Find the mean and standard deviation of $\bar{X}_n$.

(b) Let $N \sim \text{Poisson}(\nu)$, independent of the $X_i$. Find the mean and standard deviation of $\bar{X}_{N+1}$. **Hint. Use the law of total expectation, and condition on $N$.**

Question 11. Suppose we’re given a function $f(x) \geq 0$ and we want to evaluate

$$\int_{x=a}^{b} f(x) \, dx.$$ 

Here’s an approximation method: (i) draw a box that contains $f(x)$ over the range $x \in [a, b]$, (ii) scatter points uniformly at random in this box, (iii) return $A \times p$ where $A$ is the area of the box and $p$ is the fraction of points that are under the curve.

Explain why this is a special case of Monte Carlo integration. In your answer, you should identify the random variable and the function to which Monte Carlo is being applied.