Complexity Theory

Lecture 6

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http://www.cl.cam.ac.uk/teaching/1819/Complexity

Clique

Given a graph $G = (V, E)$, a subset $X \subseteq V$ of the vertices is called a *clique*, if for every $u, v \in X$, (u, v) is an edge.

As with IND, we can define a decision problem:
CLIQUE is defined as:

> *The set of pairs (G, K), where G is a graph, and K is an integer, such that G contains a clique with K or more vertices.*

Clique 2

CLIQUE is in NP by the algorithm which *guesses* a clique and then verifies it.

CLIQUE is NP-complete, since
IND $\leq_P$ CLIQUE
by the reduction that maps the pair (G, K) to $(\bar{G}, K)$, where $\bar{G}$ is the complement graph of G.

k-Colourability

A graph $G = (V, E)$ is k-colourable, if there is a function

$$\chi : V \to \{1, \ldots, k\}$$

such that, for each $u, v \in V$, if $(u, v) \in E$,

$$\chi(u) \neq \chi(v)$$

This gives rise to a decision problem for each k.
2-colourability is in P.
For all $k > 2$, k-colourability is NP-complete.

3-Colourability

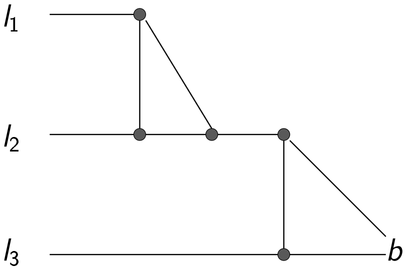
3-Colourability is in NP, as we can *guess* a colouring and verify it.

To show NP-completeness, we can construct a reduction from 3SAT to 3-Colourability.

For each variable x, we have two vertices x, $\bar{x}$ which are connected in a triangle with the vertex a (common to all variables).

In addition, for each clause containing the literals l_1, l_2 and l_3 we have a gadget.

Gadget



With a further edge from a to b.

Hamiltonian Graphs

Recall the definition of HAM—the language of Hamiltonian graphs.

Given a graph $G = (V, E)$, a *Hamiltonian cycle* in G is a path in the graph, starting and ending at the same node, such that every node in V appears on the cycle *exactly once*.

A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.

Hamiltonian Cycle

We can construct a reduction from 3SAT to HAM
Essentially, this involves coding up a Boolean expression as a graph, so that every satisfying truth assignment to the expression corresponds to a Hamiltonian circuit of the graph.

This reduction is much more intricate than the one for IND.

Travelling Salesman

Recall the travelling salesman problem

Given

- V — a set of nodes.
- $c : V \times V \to \mathbb{N}$ — a cost matrix.

Find an ordering $v_1, \ldots, v_n$ of V for which the total cost:

$$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$$

is the smallest possible.

Travelling Salesman

As with other optimisation problems, we can make a decision problem version of the Travelling Salesman problem.

The problem TSP consists of the set of triples

$$(V, c : V \times V \to \mathbb{N}, t)$$

such that there is a tour of the set of vertices V, which under the cost matrix c, has cost t or less.

Reduction

There is a simple reduction from HAM to TSP, mapping a graph (V, E) to the triple $(V, c : V \times V \to \mathbb{N}, n)$, where

$$c(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \textit{otherwise} \end{cases}$$

and n is the size of V.

Sets, Numbers and Scheduling

It is not just problems about formulas and graphs that turn out to be NP-complete.

Literally hundreds of naturally arising problems have been proved NP-complete, in areas involving network design, scheduling, optimisation, data storage and retrieval, artificial intelligence and many others.

Such problems arise naturally whenever we have to construct a solution within constraints, and the most effective way appears to be an exhaustive search of an exponential solution space.

We now examine three more NP-complete problems, whose significance lies in that they have been used to prove a large number of other problems NP-complete, through reductions.

3D Matching

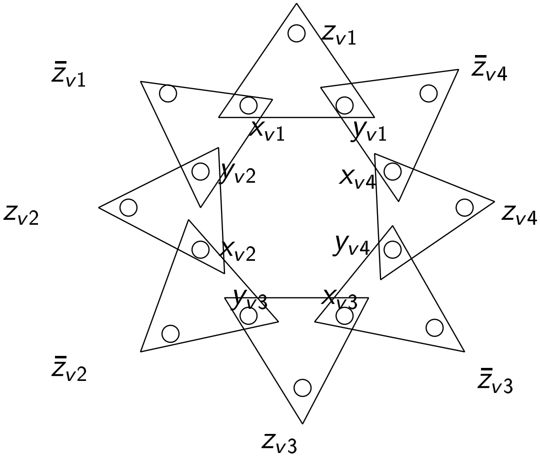
The decision problem of *3D Matching* is defined as:

> *Given three disjoint sets X, Y and Z, and a set of triples $M \subseteq X \times Y \times Z$, does M contain a matching?*
> *I.e. is there a subset $M' \subseteq M$, such that each element of X, Y and Z appears in exactly one triple of M'?*

We can show that 3DM is NP-complete by a reduction from 3SAT.

Reduction

If a Boolean expression ϕ in 3CNF has n variables, and m clauses, we construct for each variable v the following gadget.



In addition, for every clause c, we have two elements x_c and y_c.
If the literal v occurs in c, we include the triple

$$(x_c, y_c, z_{vc})$$

in M.

Similarly, if $\neg v$ occurs in c, we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

in M.
Finally, we include extra dummy elements in X and Y to make the numbers match up.