Register machines
Algorithms, informally

No precise definition of “algorithm” at the time Hilbert posed the *Entscheidungsproblem*, just examples.

Common features of the examples:

- **finite** description of the procedure in terms of elementary operations
- **deterministic** (next step uniquely determined if there is one)
- procedure may not terminate on some input data, but we can recognize when it does terminate and what the result is.
Register Machines, informally

They operate on natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots \} \) stored in (idealized) registers using the following “elementary operations”:

- add 1 to the contents of a register
- test whether the contents of a register is 0
- subtract 1 from the contents of a register if it is non-zero
- jumps ("goto")
- conditionals ("if.then_else.")
Definition. A register machine is specified by:

- finitely many registers $R_0, R_1, \ldots, R_n$ (each capable of storing a natural number);
- a program consisting of a finite list of instructions of the form $label : body$, where for $i = 0, 1, 2, \ldots$, the $(i + 1)^{th}$ instruction has label $L_i$. 
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- finitely many registers \( R_0, R_1, \ldots, R_n \) (each capable of storing a natural number);
- a program consisting of a finite list of instructions of the form \( \text{label} : \text{body} \), where for \( i = 0, 1, 2, \ldots \), the \((i + 1)^{\text{th}}\) instruction has label \( L_i \).

Instruction \textit{body} takes one of three forms:

<table>
<thead>
<tr>
<th>Instruction</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^+ \to L' )</td>
<td>add 1 to contents of register ( R ) and jump to instruction labelled ( L' )</td>
</tr>
<tr>
<td>( R^- \to L', L'' )</td>
<td>if contents of ( R ) is ( &gt; 0 ), then subtract 1 from it and jump to ( L' ), else jump to ( L'' )</td>
</tr>
<tr>
<td>HALT</td>
<td>stop executing instructions</td>
</tr>
</tbody>
</table>
Example

registers:
R₀  R₁  R₂

program:
L₀ : R₁⁻ → L₁, L₂
L₁ : R₀⁺ → L₀
L₂ : R₂⁻ → L₃, L₄
L₃ : R₀⁺ → L₂
L₄ : HALT

table:
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### Example

**registers:**
- R₀
- R₁
- R₂

**program:**
- \( L_0 : R_1^- \rightarrow L_1, L_2 \)
- \( L_1 : R_0^+ \rightarrow L_0 \)
- \( L_2 : R_2^- \rightarrow L_3, L_4 \)
- \( L_3 : R_0^+ \rightarrow L_2 \)
- \( L_4 : \text{HALT} \)

**example computation:**

<table>
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<tr>
<th>( L_i )</th>
<th>( R_0 )</th>
<th>( R_1 )</th>
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Example

registers:
\( R_0 \ R_1 \ R_2 \)

program:
\( L_0 : R_1^- \rightarrow L_1, L_2 \)
\( L_1 : R_0^+ \rightarrow L_0 \)
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Example

registers:
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L₀ : R₁⁻ → L₁, L₂
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L₄ : HALT

does not read
Example

registers:
R₀  R₁  R₂

program:
L₀ : R₁⁻ → L₁, L₂
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Example

registers:
\[ R_0 \ R_1 \ R_2 \]

program:
\[
\begin{align*}
L_0 &: R_1^- \rightarrow L_1, L_2 \\
L_1 &: R_0^+ \rightarrow L_0 \\
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\end{align*}
\]

example computation:
\[
\begin{array}{cccc}
L_i & R_0 & R_1 & R_2 \\
0 & 0 & 1 & 2 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
2 & 1 & 0 & 2 \\
3 & 1 & 0 & 1 \\
\end{array}
\]
Example

registers:
\[ R_0 \quad R_1 \quad R_2 \]

program:
\[ L_0 : R_1^- \rightarrow L_1, L_2 \]
\[ L_1 : R_0^+ \rightarrow L_0 \]
\[ L_2 : R_2^- \rightarrow L_3, L_4 \]
\[ L_3 : R_0^+ \rightarrow L_2 \]
\[ L_4 : \text{HALT} \]

data table:
\[
\begin{array}{cccc}
L_i & R_0 & R_1 & R_2 \\
0 & 0 & 1 & 2 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
2 & 1 & 0 & 2 \\
3 & 1 & 0 & 1 \\
2 & 2 & 0 & 1 \\
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L_4 &: \text{HALT}
\end{align*}

demonstration computation:
\[
\begin{array}{cccc}
L_i & R_0 & R_1 & R_2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
2 & 1 & 0 & 2 \\
3 & 1 & 0 & 1 \\
2 & 2 & 0 & 1 \\
3 & 2 & 0 & 0 \\
2 & 3 & 0 & 0
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Example

registers:
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program:
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example computation:

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</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Register machine computation

Register machine configuration:

\[ c = (\ell, r_0, \ldots, r_n) \]

where \( \ell \) = current label and \( r_i \) = current contents of \( R_i \).

**Notation:** “\( R_i = x \) [in configuration \( c \)]” means \( c = (\ell, r_0, \ldots, r_n) \) with \( r_i = x \).
Register machine computation

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Initial configurations:

\[ c_0 = (0, r_0, \ldots, r_n) \]

where \( r_i \) = initial contents of register \( R_i \).
Register machine computation

A computation of a RM is a (finite or infinite) sequence of configurations

\[ c_0, c_1, c_2, \ldots \]

where

- \( c_0 = (0, r_0, \ldots, r_n) \) is an initial configuration
- each \( c = (\ell, r_0, \ldots, r_n) \) in the sequence determines the next configuration in the sequence (if any) by carrying out the program instruction labelled \( L_\ell \) with registers containing \( r_0, \ldots, r_n \).
Halting

For a finite computation \( c_0, c_1, \ldots, c_m \), the last configuration \( c_m = (\ell, r, \ldots) \) is a halting configuration, i.e. instruction labelled \( L_\ell \) is

- either HALT (a “proper halt”)
- or \( R^+ \rightarrow L \), or \( R^- \rightarrow L, L' \) with \( R > 0 \), or
  - \( R^- \rightarrow L', L \) with \( R = 0 \)

and there is no instruction labelled \( L \) in the program (an “erroneous halt”)

L2
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E.g. \(L_0 : R_0^+ \rightarrow L_2\) \(L_1 : \text{HALT}\) halts erroneously.
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N.B. can always modify programs (without affecting their computations) to turn all erroneous halts into proper halts by adding extra HALT instructions to the list with appropriate labels.
Halting

For a finite computation $c_0, c_1, \ldots, c_m$, the last configuration $c_m = (\ell, r, \ldots)$ is a halting configuration.

Note that computations may never halt. For example,

\begin{align*}
L_0 : R_0^+ &\to L_0 \\
L_1 : \text{HALT}
\end{align*}

only has infinite computation sequences 

$(0, r), (0, r + 1), (0, r + 2), \ldots$
Graphical representation

- one node in the graph for each instruction
- arcs represent jumps between instructions
- lose sequential ordering of instructions—so need to indicate initial instruction with **START**.

<table>
<thead>
<tr>
<th>instruction</th>
<th>representation</th>
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<tbody>
<tr>
<td>$R^+ \rightarrow L$</td>
<td>$R^+ \rightarrow [L]$</td>
</tr>
<tr>
<td>$R^- \rightarrow L, L'$</td>
<td>$R^- \rightarrow [L]$</td>
</tr>
<tr>
<td>$L_0$</td>
<td>START $\rightarrow [L_0]$</td>
</tr>
<tr>
<td>HALT</td>
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registers:
\[ R_0 \quad R_1 \quad R_2 \]

program:
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\begin{align*}
L_0 & : R_1^- \rightarrow L_1, L_2 \\
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L_3 & : R_0^+ \rightarrow L_2 \\
L_4 & : \text{HALT}
\end{align*}
\]

graphical representation:

```
START
\[ \downarrow \]
\[ R_1^- \leftrightarrow R_0^+ \]
\[ \downarrow \]
\[ R_2^- \leftrightarrow R_0^+ \]
\[ \downarrow \]
\[ \text{HALT} \]
```
Example

registers:
R₀ R₁ R₂

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graphical representation:

Graphical representation is helpful for seeing what function a machine computes...
Example

Example

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<thead>
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<tr>
<td>$R_0$ $R_1$ $R_2$</td>
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graphical representation:

```
START
\downarrow
R_1^− \leftrightarrow R_0^+\downarrow
R_2^− \leftrightarrow R_0^+\downarrow
\text{HALT}
```

Claim: starting from initial configuration ($0, 0, x, y$), this machine’s computation halts with configuration ($4, x + y, 0, 0$).
Example

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Partial functions

Register machine computation is deterministic: in any non-halting configuration, the next configuration is uniquely determined by the program. So the relation between initial and final register contents defined by a register machine program is a partial function...
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Definition. A partial function from a set $X$ to a set $Y$ is specified by any subset $f \subseteq X \times Y$ satisfying

$$(x, y) \in f \land (x, y') \in f \rightarrow y = y'$$

for all $x \in X$ and $y, y' \in Y$. 
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$$(x, y) \in f \land (x, y') \in f \rightarrow y = y'$$

for all $x \in X$ and $y, y' \in Y$. 

i.e. for all $x \in X$ there is at most one $y \in Y$ with $(x, y) \in f$.

ordered pairs $\{(x, y) \mid x \in X \land y \in Y\}$
Partial functions

Notation:

- "f(x) = y" means \((x, y) \in f\)
- "f(x)↓" means \(\exists y \in Y \ (f(x) = y)\)
- "f(x)↑" means \(\neg \exists y \in Y \ (f(x) = y)\)
- \(X \rightarrow Y\) = set of all partial functions from \(X\) to \(Y\)
- \(X \rightarrow Y\) = set of all (total) functions from \(X\) to \(Y\)

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Partial functions

Notation:

- “\(f(x) = y\)” means \((x, y) \in f\)
- “\(f(x) ↓\)” means \(\exists y \in Y \ (f(x) = y)\)
- “\(f(x) ↑\)” means \(\neg \exists y \in Y \ (f(x) = y)\)
- \(X \rightarrow Y = \text{set of all partial functions from } X \text{ to } Y\)
- \(X \rightarrow Y = \text{set of all (total) functions from } X \text{ to } Y\)

**Definition.** A partial function from a set \(X\) to a set \(Y\) is **total** if it satisfies

\[f(x) ↓\]

for all \(x \in X\).
Computable functions

Definition. $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is (register machine) computable if there is a register machine $M$ with at least $n + 1$ registers $R_0, R_1, \ldots, R_n$ (and maybe more) such that for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$ and all $y \in \mathbb{N}$, the computation of $M$ starting with $R_0 = 0$, $R_1 = x_1, \ldots, R_n = x_n$ and all other registers set to 0, halts with $R_0 = y$ if and only if $f(x_1, \ldots, x_n) = y$.

Note the [somewhat arbitrary] I/O convention: in the initial configuration registers $R_1, \ldots, R_n$ store the function’s arguments (with all others zeroed); and in the halting configuration register $R_0$ stores its value (if any).
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N.B. there may be many different $M$ that compute the same partial function $f$. 
Example

registers:
\[ R_0 \quad R_1 \quad R_2 \]

program:
\[
\begin{align*}
L_0 & : R_1^- \rightarrow L_1, L_2 \\
L_1 & : R_0^+ \rightarrow L_0 \\
L_2 & : R_2^- \rightarrow L_3, L_4 \\
L_3 & : R_0^+ \rightarrow L_2 \\
L_4 & : \text{HALT}
\end{align*}
\]

graphical representation:

Claim: starting from initial configuration \((0, 0, x, y)\), this machine’s computation halts with configuration \((4, x + y, 0, 0)\). So \(f(x, y) \triangleq x + y\) is computable.
Multiplication $f(x, y) \triangleq xy$ is computable
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Multiplication $f(x, y) \triangleq xy$ is computable

\[ (R_0, R_2, R_3) := (R_0 + R_2, 0, R_2 + R_3) \]

\[ (R_{21} R_3) := (R_2 + R_3, 0) \]
Multiplication $f(x, y) \triangleq xy$ is computable
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If the machine is started with $(R_0, R_1, R_2, R_3) = (0, x, y, 0)$, it halts with $(R_0, R_1, R_2, R_3) = (xy, 0, y, 0)$. 
Further examples

The following arithmetic functions are all computable. (Proof—left as an exercise!)

projection:  $p(x, y) \triangleq x$

constant:  $c(x) \triangleq n$

truncated subtraction:  $x \cdot y \triangleq \begin{cases} x - y & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases}$
Further examples

The following arithmetic functions are all computable. (Proof—left as an exercise!)

**integer division:**

\[ x \div y \triangleq \begin{cases} \text{integer part of } x/y & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases} \]

**integer remainder:**

\[ x \mod y \triangleq x - y \cdot (x \div y) \]

**exponentiation base 2:**

\[ e(x) \triangleq 2^x \]

**logarithm base 2:**

\[ \log_2(x) \triangleq \begin{cases} \text{greatest } y \text{ such that } 2^y \leq x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \]
W.l.o.g. can use RMs with only one HALT

Sequential composition $M_1; M_2$

IF $R = 0$ THEN $M_1$ ELSE $M_2$

WHILE $R \neq 0$ DO $M$

N.B. interference