Iteration 1:

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations

Demonstration: Solving TSP via LP
Iteration 1: Eliminate Subtour 1, 2, 41, 42

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations
Iteration 1: Eliminate Subtour 1, 2, 41, 42

Objective value: \(-641.000000\), 861 variables, 945 constraints, 1809 iterations

Disallow subtour (1, 2, 42, 41) by adding this constraint to the LP:

\[ x(2, 1) + x(41, 1) + x(42, 1) + x(41, 2) + x(42, 2) + x(42, 41) \leq 3 \]
Iteration 1: Eliminate Subtour 1, 2, 41, 42

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations

Disallow subtour (1, 2, 42, 41) by adding this constraint to the LP:

$$x(2, 1) + x(41, 1) + x(42, 1) + x(41, 2) + x(42, 2) + x(42, 41) \leq 3$$

Equivalent to: $S = \{1, 2, 41, 42\}$,

$$\sum_{i \in S, j \in V \setminus S} x(\max(i, j), \min(i, j)) \geq 2$$
Iteration 2:

Objective value: $-676.000000$, 861 variables, 946 constraints, 1802 iterations
Iteration 2: Eliminate Subtour $3 - 9$

Objective value: $-676.000000$, 861 variables, 946 constraints, 1802 iterations
Iteration 3:

Objective value: $-681.000000$, 861 variables, 947 constraints, 1984 iterations
Iteration 3: Eliminate Subtour 24, 25, 26, 27

Objective value: $-681.000000$, 861 variables, 947 constraints, 1984 iterations
Iteration 4:

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations
Iteration 4: Eliminate Cut 11 – 23

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations
Iteration 4: Eliminate Cut 11 – 23

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations

Tour has to include at least two edges between $S = \{11, 12, \ldots, 23\}$ and $V \setminus S$:

$$\sum_{i \in S,j \in V \setminus S} x(\max(i,j), \min(i,j)) \geq 2.$$
Iteration 5:

Objective value: $-686.000000$, 861 variables, 949 constraints, 2446 iterations
Iteration 5: Eliminate Subtour 13 – 23

Objective value: $-686.000000$, 861 variables, 949 constraints, 2446 iterations
Iteration 6:

Objective value: \(-694.500000\), 861 variables, 950 constraints, 1690 iterations
Iteration 6: Eliminate Cut 13 – 17

Objective value: $-694.500000$, 861 variables, 950 constraints, 1690 iterations
Iteration 7:

Objective value: $-697.000000$, 861 variables, 951 constraints, 2212 iterations
Iteration 7: Branch 1a $x_{18,15} = 0$

Objective value: $-697.000000$, 861 variables, 951 constraints, 2212 iterations
Iteration 8:

Objective value: $-698.000000$, 861 variables, 952 constraints, 1878 iterations
Iteration 8: Branch 2a $x_{17,13} = 0$

Objective value: $-698.000000$, 861 variables, 952 constraints, 1878 iterations
Iteration 9:

Objective value: $-699.000000$, 861 variables, 953 constraints, 2281 iterations
Iteration 9: Branch 2b $x_{17,13} = 1$

Objective value: $-699.000000$, 861 variables, 953 constraints, 2281 iterations
Iteration 10:

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations
Iteration 10:

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations

Branch & Bound procedure would stop here, since value of the best LP solution for $x_{18,15} = 0$ is worse than a previously found tour.
Iteration 10: Branch 1b $x_{18,15} = 1$

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations

Branch & Bound procedure would stop here, since value of the best LP solution for $x_{18,15} = 0$ is worse than a previously found tour.
Iteration 11:

Objective value: $-701.000000$, 861 variables, 953 constraints, 2506 iterations
Iteration 11: Branch & Bound terminates

Objective value: $-701.000000$, 861 variables, 953 constraints, 2506 iterations
Branch & Bound Overview

1: LP solution 641

Eliminate Subtour 1, 2, 41, 42

Eliminate Subtour 3 − 9

Eliminate Subtour 24 − 25, 26, 27

Eliminate Cut 11 − 23

Eliminate Subtour 10 − 11, 12

Eliminate Cut 13 − 17, 18 = 0

x 17, 13 = 0

x 17, 13 = 1

Cut branch, since LP solution worse than current best possible tour.

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641

Eliminate Subtour 1, 2, 41, 42

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641

Eliminate Subtour 1, 2, 41, 42

2: LP solution 676

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
Branch & Bound Overview

1: LP solution 641
   → Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   → Eliminate Subtour 3 – 9
3: LP solution 681
   → Eliminate Subtour 24, 25, 26, 27

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686

Eliminate Subtour 10, 11, 12

Eliminate Cut 13 – 17, 18, 15

x_{17}, x_{13} = 0

x_{17}, x_{15} = 1

Cut branch, since LP solution worse than current best possible tour.
Branch & Bound Overview

1: LP solution 641
- Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
- Eliminate Subtour 3 – 9

3: LP solution 681
- Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
- Eliminate Cut 11 – 23

5: LP solution 686
- Eliminate Subtour 10, 11, 12

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   - Eliminate Subtour 3 – 9

3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   - Eliminate Cut 11 – 23

5: LP solution 686
   - Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   - Cut branch, since LP solution worse than current best possible tour.

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

Demonstration: Solving TSP via LP
Demonstration: Solving TSP via LP

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697
   \[ x_{18,15} = 0 \]
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697
8: LP solution 698

\[ x_{18,15} = 0 \]

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   → Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   → Eliminate Subtour 3 – 9

3: LP solution 681
   → Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   → Eliminate Cut 11 – 23

5: LP solution 686
   → Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   → Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   - $x_{18,15} = 0$
   - $x_{17,13} = 0$

9: Valid tour 699

10: LP solution 700

11: Valid tour 701

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   \[ x_{18,15} = 0 \]

9: Valid tour 699
   \[ x_{17,13} = 0 \]
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   $x_{18,15} = 0$

9: Valid tour 699
   $x_{17,13} = 0$

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697

x_{18,15} = 0
x_{17,13} = 1

8: LP solution 698
9: Valid tour 699
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   \[ x_{18,15} = 0 \]
   \[ x_{17,13} = 0 \]

9: Valid tour 699

10: LP solution 700
   \[ x_{17,13} = 1 \]
   Cut branch, since LP solution worse than current best possible tour.
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697
8: LP solution 698
   \( x_{18,15} = 0 \)
9: Valid tour 699
   \( x_{17,13} = 0 \)
10: LP solution 700
   \( x_{17,13} = 1 \)

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697
   \[ x_{18,15} = 0 \]

8: LP solution 698
   \[ x_{17,13} = 0, \quad x_{17,13} = 1 \]

9: Valid tour 699

10: LP solution 700

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   \[ x_{18,15} = 0 \]
   \[ x_{18,15} = 1 \]
   \[ x_{17,13} = 0 \]
   \[ x_{17,13} = 1 \]

9: Valid tour 699

10: LP solution 700

Demonstration: Solving TSP via LP
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   \[ x_{18, 15} = 0 \]
   \[ x_{17, 13} = 0 \]

9: Valid tour 699
   \[ x_{17, 13} = 1 \]

10: LP solution 700
   \[ x_{18, 15} = 1 \]

11: Valid tour 701
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   \( x_{18,15} = 0 \)
   \( x_{17,13} = 0 \)

9: Valid tour 699

10: LP solution 700
    \( x_{17,13} = 1 \)
    \( x_{18,15} = 1 \)

11: Valid tour 701
Iteration 8: Objective 697
What about choosing a different branching variable?
Solving Progress (Alternative Branch 1)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697
Solving Progress (Alternative Branch 1)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3

3: LP solution 681
   Eliminate Subtour 4

4: LP solution 682.5
   Eliminate Cut 5

5: LP solution 686
   Eliminate Subtour 6

6: LP solution 686
   Eliminate Subtour 7

7: LP solution 688
   Eliminate Subtour 8

8: LP solution 697
   Eliminate Subtour 9

9: $x_{15,18} = 1$

10: $x_{15,18} = 0$
Alternative Branch 1: $x_{18,15}$, Objective 697
Alternative Branch 1: \( x_{18,15} \), Objective 697
Alternative Branch 1a: $x_{18,15} = 1$, Objective 701 (Valid Tour)
Alternative Branch 1b: $x_{18,15} = 0$, Objective 698
Solving Progress (Alternative Branch 1)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 13 – 17
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 686
   Eliminate Subtour 13 – 23
7: LP solution 688
   Eliminate Subtour 11 – 23
8: LP solution 697
   \[ x_{18,15} = 1 \]
   \[ x_{18,15} = 0 \]
9: valid tour 701
10: LP solution 698
Solving Progress (Alternative Branch 2)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697
Solving Progress (Alternative Branch 2)

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   - Eliminate Subtour 3 – 9
3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   - Eliminate Cut 13 – 17
5: LP solution 686
   - Eliminate Subtour 10, 11, 12
6: LP solution 686
   - Eliminate Subtour 13 – 23
7: LP solution 688
   - Eliminate Subtour 11 – 23
8: LP solution 697
   - $x_{27,22} = 1$
   - $x_{27,22} = 0$
9: ???
10: ???
Demonstration: Solving TSP via LP
Alternative Branch 2: $x_{27,22}$, Objective 697

Demonstration: Solving TSP via LP
Alternative Branch 2a: \( x_{27,22} = 1 \), Objective 708 (Valid tour)
Alternative Branch 2b: $x_{27,22} = 0$, Objective 697.75
Solving Progress (Alternative Branch 2)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697

9: valid tour 708

10: LP solution 697.75

\[ x_{27,22} = 1 \]
\[ x_{27,22} = 0 \]
Solving Progress (Alternative Branch 3)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697
Eliminate Subtour 1, 2, 41, 42
Eliminate Subtour 3 – 9
Eliminate Subtour 24, 25, 26, 27
Eliminate Cut 13 – 17
Eliminate Subtour 10, 11, 12
Eliminate Subtour 13 – 23
Eliminate Subtour 11 – 23

8: LP solution 697
9: ???
10: ???
Alternative Branch 3: $x_{27,24}$, Objective 697
Alternative Branch 3: $x_{27,24}$, Objective 697

Demonstration: Solving TSP via LP
Alternative Branch 3a: $x_{27,24} = 1$, Objective 697.75
Alternative Branch 3b: $x_{27,24} = 0$, Objective 698
Solving Progress (Alternative Branch 3)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 13 – 17
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 686
   Eliminate Subtour 13 – 23
7: LP solution 688
   Eliminate Subtour 11 – 23
8: LP solution 697

\[ x_{27,24} = 1 \]
\[ x_{27,24} = 0 \]

9: LP solution 697.75
10: LP solution 698

Not only do we have to explore (and branch further in) both subtrees, but also the optimal tour is in the subtree with larger LP solution!
Solving Progress (Alternative Branch 3)

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   - Eliminate Subtour 3 – 9

3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5

Not only do we have to explore (and branch further in) both subtrees, but also the optimal tour is in the subtree with larger LP solution!

5: LP solution 686
   - Eliminate Cut 13 – 17

6: LP solution 688
   - Eliminate Subtour 10, 11, 12

7: LP solution 688
   - Eliminate Subtour 13 – 23

8: LP solution 697
   - Eliminate Subtour 11 – 23
   - $x_{27,24} = 1$
   - $x_{27,24} = 0$

9: LP solution 697.75
10: LP solution 698
How can one generate these constraints automatically?
Conclusion (1/2)

- How can one generate these constraints automatically?
  - **Subtour Elimination**: Finding Connected Components
  - **Small Cuts**: Finding the Minimum Cut in Weighted Graphs
Conclusion (1/2)

- How can one generate these constraints automatically?
  - **Subtour Elimination:** Finding Connected Components
  - **Small Cuts:** Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
  There are exponentially many of them!
Conclusion (1/2)

- How can one generate these constraints automatically?
  **Subtour Elimination**: Finding Connected Components
  **Small Cuts**: Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?
Conclusion (1/2)

- **How can one generate these constraints automatically?**
  - **Subtour Elimination:** Finding Connected Components
  - **Small Cuts:** Finding the Minimum Cut in Weighted Graphs

- **Why don’t we add all possible Subtour Elimination constraints to the LP?**
  There are exponentially many of them!

- **Should the search tree be explored by BFS or DFS?**
  - **BFS** may be more attractive, even though it might need more memory.
Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?
  BFS may be more attractive, even though it might need more memory.

CONCLUDING REMARK

It is clear that we have left unanswered practically any question one might pose of a theoretical nature concerning the traveling-salesman problem; however, we hope that the feasibility of attacking problems involving a moderate number of points has been successfully demonstrated, and that perhaps some of the ideas can be used in problems of similar nature.
Conclusion (2/2)

- Eliminate Subtour 1, 2, 41, 42
- Eliminate Subtour 3 – 9
- **Eliminate Subtour 10, 11, 12**
- Eliminate Subtour 11 – 23
- Eliminate Subtour 13 – 23
- Eliminate Cut 13 – 17
- Eliminate Subtour 24, 25, 26, 27
Conclusion (2/2)

- Eliminate Subtour 1, 2, 41, 42
- Eliminate Subtour 3 – 9
- **Eliminate Subtour 10, 11, 12**
- Eliminate Subtour 11 – 23
- Eliminate Subtour 13 – 23
- Eliminate Cut 13 – 17
- Eliminate Subtour 24, 25, 26, 27

THE 49-CITY PROBLEM*

The optimal tour $\bar{x}$ is shown in Fig. 16. The proof that it is optimal is given in Fig. 17. To make the correspondence between the latter and its programming problem clear, we will write down in addition to 42 relations in non-negative variables (2), a set of 25 relations which suffice to prove that $D(x)$ is a minimum for $\bar{x}$. We distinguish the following subsets of the 42 cities:

$$
S_1 = \{1, 2, 41, 42\} \quad S_5 = \{13, 14, \ldots, 23\} \\
S_2 = \{3, 4, \ldots, 9\} \quad S_6 = \{13, 14, 15, 16, 17\} \\
S_3 = \{1, 2, \ldots, 9, 29, 30, \ldots, 42\} \quad S_7 = \{24, 25, 26, 27\} \\
S_4 = \{11, 12, \ldots, 23\} 
$$