Consider the following Natural Deduction rule:

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\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{(Impl)}
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Rule can be read in two ways:

- Backwards: “to prove $\Gamma \vdash \phi \rightarrow \psi$ suffices to prove $\Gamma, \phi \vdash \psi$”
- Forwards: “if $\Gamma, \phi \vdash \psi$ holds then $\Gamma \vdash \phi \rightarrow \psi$ holds”
Forward vs. backward proof

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- Backwards: “to prove \( \Gamma \vdash \phi \rightarrow \psi \) suffices to prove \( \Gamma, \phi \vdash \psi \)”
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With apply-style proofs we are reasoning backwards

Decomposing complex goals into simpler goals

Using a backward reading of rules and theorems
Meta vs. object-level

Consider the same Natural Deduction rule:

$$
\Gamma, \phi \vdash \psi \\
\frac{}{\Gamma \vdash \phi \to \psi} \quad \text{(Impl)}
$$

Recall the forward reading: "if \( \Gamma, \phi \vdash \psi \) holds then \( \Gamma \vdash \phi \to \psi \) holds"

Note there are two different kinds of implication at play here!

The implication in the logic we are reasoning about: \( \phi \to \psi \)

The implication in our meta-language, informal English: "if-then"

Isabelle uses a logic rather than informal English for this purpose

The "fat arrow" = \( \Rightarrow \) replaces the English "if-then" in Isabelle
Consider the same Natural Deduction rule:

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\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{(Impl)}
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Recall the forward reading: “if \( \Gamma, \phi \vdash \psi \) holds then \( \Gamma \vdash \phi \rightarrow \psi \) holds”
Consider the same Natural Deduction rule:

\[
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{(ImpI)}
\]

Recall the forward reading: “if \( \Gamma, \phi \vdash \psi \) holds then \( \Gamma \vdash \phi \rightarrow \psi \) holds”

Note there are \textit{two} different kinds of implication at play here!

The implication in the logic we are reasoning about: \( \phi \rightarrow \psi \)

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Consider the same Natural Deduction rule:

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\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad (\text{Impl})
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Recall the forward reading: “if \( \Gamma, \phi \vdash \psi \) holds then \( \Gamma \vdash \phi \rightarrow \psi \) holds”

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The implication in the logic we are reasoning about: \( \phi \rightarrow \psi \)

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The “fat arrow” \( \implies \) replaces the English “if-then” in Isabelle
Moreover...

In the rule:

\[ \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{(ImpI)} \]

...there’s also hidden universal quantification
Moreover...

In the rule:

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{(ImpI)}$$

...there's also hidden universal quantification

“*For every* $\Gamma$, $\phi$, and $\psi$: if $\Gamma, \phi \vdash \psi$ holds then $\Gamma \vdash \phi \rightarrow \psi$ holds”
Moreover...

In the rule:

\[
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{(ImpI)}
\]

...there's also hidden universal quantification

“For every \( \Gamma, \phi, \) and \( \psi \): if \( \Gamma, \phi \vdash \psi \) holds then \( \Gamma \vdash \phi \rightarrow \psi \) holds”

The meta-level universal quantifier (\( \bigwedge \)) replaces the English “for-every” in Isabelle

Thus the Natural Deduction rule above would be rendered as

\[
\bigwedge \Gamma \phi \psi. \quad \Gamma, \phi \vdash \psi \implies \Gamma \vdash \phi \rightarrow \psi
\]

when embedded in HOL