# L11: Algebraic Path Problems with applications to Internet Routing Lectures 5 and 6 An introduction to Combinators for Algebraic Systems (CAS)

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## Combinators for Algebraic Structures (CAS)

**Basic idea**

Instead of proving all the required properties for a defined algebraic structure, we will define a language of combinators in which we can simultaneously defines an algebraic structure and then automatically compute, for a fixed set of properties $\mathbb{P}$, which of these properties holds.

For every $n$-ary combinator $C$,

$$\forall i \in \{1,\ 2, \cdots, n\},\ \forall P \in \mathbb{P},\ P(E_i) \vee \neg P(E_i)$$
$$\Longrightarrow$$
$$\forall P \in \mathbb{P},\ P(C(E_1,\ E_2,\ \cdots E_n)) \vee \neg P(C(E_1,\ E_2,\ \cdots E_n))$$

We will be working in **constructive logic** where $P \vee \neg P$ is **not** an axiom!

# Base sets of properties

## For Semigroups, $\mathbb{P}_0^{SG}$

$$
\begin{array}{rcl}
\mathbb{AS}(S,\ \bullet) & \equiv & \forall a, b, c \in S,\ a \bullet (b \bullet c) = (a \bullet b) \bullet c \\
\mathbb{ID}(S,\ \bullet) & \equiv & \exists \alpha \in S,\ \mathbb{IID}(S,\ \bullet,\ \alpha) \\
\mathbb{AN}(S,\ \bullet) & \equiv & \exists \omega \in S,\ \mathbb{IAN}(S,\ \bullet,\ \omega) \\
\mathbb{CM}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b = b \bullet a \\
\mathbb{SL}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b \in \{a,\ b\} \\
\mathbb{IP}(S,\ \bullet) & \equiv & \forall a \in S,\ a \bullet a = a
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{IID}(S,\ \bullet,\ \alpha) & \equiv & \alpha \in S \wedge \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{IAN}(S,\ \bullet,\ \omega) & \equiv & \omega \in S \wedge \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega
\end{array}
$$



# Base sets of properties

## For Bi-Semigroup, $\mathbb{P}_0^{BS}$

$\mathbb{P}_0^{BS}$

$$
\begin{array}{rcl}
\mathbb{ZA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{0} \in S,\ \mathbb{IID}(S,\ \oplus,\ \overline{0}) \wedge \mathbb{IAN}(S,\ \otimes,\ \overline{0}) \\
\mathbb{OA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{1} \in S,\ \mathbb{IID}(S,\ \otimes,\ \overline{1}) \wedge \mathbb{IAN}(S,\ \oplus,\ \overline{1}) \\
\mathbb{LD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \\
\mathbb{RD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)
\end{array}
$$

Why are we starting with $\mathbb{P}_0^{SG}$ and $\mathbb{P}_0^{BS}$ instead of some other properties? Because we want to solve matrix equations with the algebraic structures that we can define in CAS.

# Add identity

$$\mathrm{AddId}(\alpha,\ (S,\ \bullet)) \equiv (S \uplus \{\alpha\}, \bullet_{\alpha}^{\mathrm{id}})$$

where

$$a \bullet_{\alpha}^{\mathrm{id}} b \ \equiv \begin{cases} a & (\text{if } b = \mathrm{inr}(\alpha)) \\ b & (\text{if } a = \mathrm{inr}(\alpha)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

## disjoint union

$$A \uplus B \equiv \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$$



# Add identity

## Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{AN}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \\ \mathbb{CM}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

# Inserting an annihilator

$$\text{AddAn}(\omega,\ (S,\ \bullet)) \equiv (S \uplus \{\omega\}, \bullet_{\omega}^{\text{an}})$$

where

$$a \bullet_{\omega}^{\text{an}} b \quad \equiv \quad \begin{cases} \text{inr}(\omega) & (\text{if } b = \text{inr}(\omega)) \\ \text{inr}(\omega) & (\text{if } a = \text{inr}(\omega)) \\ \text{inl}(x \bullet y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$



# Add annihilator

## Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\text{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\text{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \\ \mathbb{AN}(\text{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{CM}(\text{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\text{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\text{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

## Operations for adding a zero, a one

$$\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes)) \equiv (S \uplus \{\overline{0}\},\ \oplus_{0}^{\text{id}},\ \otimes_{0}^{\text{an}})$$

$$\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes)) \equiv (S \uplus \{\overline{1}\},\ \oplus_{1}^{\text{an}},\ \otimes_{1}^{\text{id}})$$

$$a \bullet_{\alpha}^{\text{id}} b \equiv \begin{cases} a & (\text{if } b = \text{inr}(\alpha)) \\ b & (\text{if } a = \text{inr}(\alpha)) \\ \text{inl}(x \bullet y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

$$a \bullet_{\omega}^{\text{an}} b \equiv \begin{cases} \text{inr}(\omega) & (\text{if } b = \text{inr}(\omega)) \\ \text{inr}(\omega) & (\text{if } a = \text{inr}(\omega)) \\ \text{inl}(x \bullet y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$



## Property management for AddZero

### Easy Exercises

$$\begin{array}{lcl} \mathbb{LD}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \\ \mathbb{RD}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \\ \mathbb{ZA}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{OA}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus,\ \otimes) \end{array}$$

# Why Easy Exercises?

## Consider left distributivity ($\mathbb{LD}$)

| $a$ | $b$ | $c$ | $a \otimes_0^{an} (b \oplus_0^{id} c)$ | $(a \otimes_0^{an} b) \oplus_0^{id} (a \otimes_0^{an} c)$ |
|---|---|---|---|---|
| $\mathrm{inl}(a')$ | $\mathrm{inl}(b')$ | $\mathrm{inl}(c')$ | $\mathrm{inl}(a' \otimes (b' \oplus c'))$ | $\mathrm{inl}((a' \otimes b') \oplus (a' \otimes c'))$ |
| $\mathrm{inr}(\overline{0})$ | $\mathrm{inl}(b')$ | $\mathrm{inl}(c')$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ |
| $\mathrm{inl}(a')$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inl}(c')$ | $\mathrm{inl}(a' \oplus c')$ | $\mathrm{inl}(a' \oplus c')$ |
| $\mathrm{inl}(a')$ | $\mathrm{inl}(b')$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inl}(a' \oplus b')$ | $\mathrm{inl}(a' \oplus b')$ |
| $\mathrm{inl}(a')$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ |
| $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ |



# However, adding a one is more complicated!

## Consider left distributivity ($\mathbb{LD}$)

| $a$ | $b$ | $c$ | $a \otimes_1^{id} (b \oplus_1^{an} c)$ | $(a \otimes_1^{id} b) \oplus_1^{an} (a \otimes_1^{id} c)$ |
|---|---|---|---|---|
| $\mathrm{inl}(a')$ | $\mathrm{inl}(b')$ | $\mathrm{inl}(c')$ | $\mathrm{inl}(a' \otimes (b' \oplus c'))$ | $\mathrm{inl}((a' \otimes b') \oplus (a' \otimes c'))$ |
| $\mathrm{inr}(\overline{1})$ | $\mathrm{inl}(b')$ | $\mathrm{inl}(c')$ | $\mathrm{inl}(b' \oplus c')$ | $\mathrm{inl}(b' \oplus c')$ |
| $\mathrm{inl}(a')$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inl}(c')$ | $\mathrm{inl}(a')$ | $\mathrm{inl}((a' \oplus (a' \otimes c'))$ |
| $\mathrm{inl}(a')$ | $\mathrm{inl}(b')$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inl}(a')$ | $\mathrm{inl}((a' \otimes b') \oplus a')$ |
| $\mathrm{inl}(a')$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inl}(a')$ | $\mathrm{inl}(a' \oplus a')$ |
| $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ |

## What is this?

### $a = (a \otimes b) \oplus a$

Suppose $\oplus$ is idempotent and commutative and we let $a \leqslant b \equiv a = a \oplus b$. We know that

$$b \leqslant c \Rightarrow a \otimes b \leqslant a \otimes c$$

since $b = b \oplus c$ implies $a \otimes b = a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$. That is $\otimes$ is order preserving.
Now $a = (a \otimes b) \oplus a$ is telling us something else, that

$$a \leqslant a \otimes b.$$

That is, that multiplication is inflationary.



## Absorption

### $\mathbb{AB}$sorption properties (name is from lattice theory)

$$\begin{array}{lcl} \mathbb{RAB}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (a \otimes b) \oplus a = a \oplus (a \otimes b) \\ \mathbb{LAB}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (b \otimes a) \oplus a = a \oplus (b \otimes a) \end{array}$$

If we want to **close** our simple language of combinators $\{\mathrm{AddZero},\ \mathrm{AddOne}\}$ with respect to $\mathbb{P}_0^{SG}$ and $\mathbb{P}_0^{BS}$, we are forced to add $\{\mathbb{RAB},\ \mathbb{LAB}\}$.

## Rules for absorption for $\mathrm{AddZero}$? Consider $\mathbb{RAB}$

**AddZero**

| $a$ | $b$ | $(a \otimes_{\overline{0}}^{\text{an}} b) \oplus_{\overline{0}}^{\text{id}} a$ | $a \oplus_{\overline{0}}^{\text{id}} (a \otimes_{\overline{0}}^{\text{an}} b)$ |
|---|---|---|---|
| $\mathrm{inl}(a')$ | $\mathrm{inl}(b')$ | $\mathrm{inl}((a' \otimes b') \oplus a)$ | $\mathrm{inl}(a' \oplus (a' \otimes b'))$ |
| $\mathrm{inr}(\overline{0})$ | $\mathrm{inl}(b')$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ |
| $\mathrm{inl}(a')$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inl}(a')$ | $\mathrm{inl}(a')$ |
| $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ | $\mathrm{inr}(\overline{0})$ |

$$\mathbb{RAB}(\mathrm{AddZero}(\overline{0}, (S, \oplus, \otimes))) \Leftrightarrow \mathbb{RAB}(S, \oplus, \otimes)$$
$$\mathbb{LAB}(\mathrm{AddZero}(\overline{0}, (S, \oplus, \otimes))) \Leftrightarrow \mathbb{LAB}(S, \oplus, \otimes)$$



## Rules for absorption for $\mathrm{AddOne}$? Consider $\mathbb{RAB}$

**AddOne**

| $a$ | $b$ | $(a \otimes_{\overline{1}}^{\text{id}} b) \oplus_{\overline{1}}^{\text{an}} a$ | $a \oplus_{\overline{1}}^{\text{an}} (a \otimes_{\overline{1}}^{\text{id}} b)$ |
|---|---|---|---|
| $\mathrm{inl}(a')$ | $\mathrm{inl}(b')$ | $\mathrm{inl}((a' \otimes b') \oplus a)$ | $\mathrm{inl}(a' \oplus (a' \otimes b'))$ |
| $\mathrm{inr}(\overline{1})$ | $\mathrm{inl}(b')$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ |
| $\mathrm{inl}(a')$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inl}(a')$ | $\mathrm{inl}(a' \oplus a')$ |
| $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ | $\mathrm{inr}(\overline{1})$ |

# Property management for AddOne

$$\begin{array}{rcl}
\mathbb{LD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{RD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{LAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{ZA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus,\ \otimes) \\
\mathbb{OA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{RAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{LAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus)
\end{array}$$



# Direct Product of Semigroups

Let $(S,\ \bullet)$ and $(T,\ \diamond)$ be semigroups.

**Definition (Direct product semigroup)**

The direct product is denoted

$$(S,\ \bullet) \times (T,\ \diamond) \equiv (S \times T,\ \star)$$

where

$$\star = \bullet \times \diamond$$

is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = (s_1 \bullet s_2,\ t_1 \diamond t_2).$$

### Easy exercises

$$\begin{array}{rcl}
\mathbb{AS}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{AS}(S,\bullet)\wedge\mathbb{AS}(T,\diamond)\\
\mathbb{ID}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{ID}(S,\bullet)\wedge\mathbb{ID}(T,\diamond)\\
\mathbb{AN}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{AN}(S,\bullet)\wedge\mathbb{AN}(T,\diamond)\\
\mathbb{CM}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{CM}(S,\bullet)\wedge\mathbb{CM}(T,\diamond)\\
\mathbb{IP}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{IP}(S,\bullet)\wedge\mathbb{IP}(T,\diamond)
\end{array}$$

### What about $\mathbb{SL}$?

Consider the product of two selective semigroups, such as $(\mathbb{N},\ \min)\times(\mathbb{N},\ \max)$.

$$(10,\ 10)\star(1,\ 3)=(1,\ 10)\notin\{(10,\ 10),\ (1,\ 3)\}$$

The result in this case is not selective!



## Direct product and $\mathbb{SL}$?

$$\mathbb{SL}((S,\bullet)\times(T,\diamond)) \Leftrightarrow (\mathbb{IR}(S,\bullet)\wedge\mathbb{IR}(T,\diamond))\vee(\mathbb{IL}(S,\bullet)\wedge\mathbb{IL}(T,\diamond))$$

$$\begin{array}{rcl}
\mathbb{IR}\ \text{is right} & \equiv & \forall s,t\in S,\ s\bullet t=t\\
\mathbb{IL}\ \text{is left} & \equiv & \forall s,t\in S,\ s\bullet t=s
\end{array}$$

Adding direct product to our semigroup combinators forces us to add $\mathbb{IR}$ and $\mathbb{IR}$ to our properties.

## Revisit all combinators seen so far ...

$$
\begin{array}{rcl}
\mathbb{IR}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\\
\mathbb{IR}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\\
\mathbb{IR}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond) \\
\mathbb{IL}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond)
\end{array}
$$



## Lexicographic Product of Semigroups

### Lexicographic product semigroup

Suppose that semigroup $(S,\ \bullet)$ is commutative, and selective and that $(T,\ \diamond)$ is a semigroup.

$$(S,\ \bullet)\ \vec{\times}\ (T,\ \diamond) \equiv (S \times T,\ \star)$$

where $\star \equiv \bullet\ \vec{\times}\ \diamond$ is defined as

$$
(s_1,\ t_1) \star (s_2,\ t_2) = \begin{cases}
(s_1 \bullet s_2,\ t_1 \diamond t_2) & s_1 = s_1 \bullet s_2 = s_2 \\
(s_1 \bullet s_2,\ t_1) & s_1 = s_1 \bullet s_2 \neq s_2 \\
(s_1 \bullet s_2,\ t_2) & s_1 \neq s_1 \bullet s_2 = s_2
\end{cases}
$$

## Examples

**$(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \min)$**

$$
\begin{array}{rcl}
(1,\ 17) \star (2,3) & = & (1,17) \\
(2,\ 17) \star (2,3) & = & (2,3) \\
(2,\ 3) \star (2,3) & = & (2,3)
\end{array}
$$

**$(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \max)$**

$$
\begin{array}{rcl}
(1,\ 17) \star (2,3) & = & (1,17) \\
(2,\ 17) \star (2,3) & = & (2,17) \\
(2,\ 3) \star (2,3) & = & (2,3)
\end{array}
$$

**$(\mathbb{N},\ \max)\ \vec{\times}\ (\mathbb{N},\ \min)$**

$$
\begin{array}{rcl}
(1,\ 17) \star (2,3) & = & (2,3) \\
(2,\ 17) \star (2,3) & = & (2,3) \\
(2,\ 3) \star (2,3) & = & (2,3)
\end{array}
$$



**Assuming $\mathbb{AS}(S, \bullet) \wedge \mathbb{CM}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$**

$$
\begin{array}{rcl}
\mathbb{AS}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{IP}(T, \diamond) \\
\mathbb{SL}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{SL}(T, \diamond) \\
\mathbb{IR}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

All easy, except for $\mathbb{AS}$! We are assuming commutativity and selectivity in order to guarantee associativity.

## Lexicographic product for Bi-Semigroups

**Assume $\mathbb{AS}(S, \oplus_S) \wedge \mathbb{AS}(T, \oplus_T) \wedge \mathbb{CM}(S, \oplus_S) \wedge \mathbb{SL}(S, \oplus_S)$**

Let

$$(S, \oplus_S, \otimes_S) \vec{\times} (T, \oplus_T, \otimes_T) \equiv (S \times T, \oplus_S \vec{\times} \oplus_T, \otimes_S \times \otimes_T)$$

That is, the additive component is a lexicographic product, and the multiplicative component is a direct product.



## Examples

**$\oplus = \min \vec{\times} \max$, $\otimes = + \times \min$**

$$\begin{aligned}(3, 10) \otimes ((17, 21) \oplus (11, 4)) &= (3, 10) \otimes (11, 4) \\ &= (14, 4)\end{aligned}$$

$$\begin{aligned}((3, 10) \otimes (17, 21)) \oplus ((3, 10) \otimes (11, 4)) &= (20, 10) \oplus (14, 4) \\ &= (14, 4)\end{aligned}$$

**$\oplus = \max \vec{\times} \min$, $\otimes = \min \times +$**

$$\begin{aligned}(3, 10) \otimes ((17, 21) \oplus (11, 4)) &= (3, 10) \otimes (17, 21) \\ &= (3, 31)\end{aligned}$$

$$\begin{aligned}((3, 10) \otimes (17, 21)) \oplus ((3, 10) \otimes (11, 4)) &= (3, 31) \oplus (3, 14) \\ &= (3, 14)\end{aligned}$$

## Distributivity?

**Theorem: If $\oplus_S$ is commutative and selective, then**

$$\mathbb{LD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T))\ \Leftrightarrow$$
$$\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T))$$

$$\mathbb{RD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T))\ \Leftrightarrow$$
$$\mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{RC}(S,\ \otimes_S) \vee \mathbb{RK}(T,\ \otimes_T))$$

**Left and Right Cancellative**

$$\mathbb{LC}(X,\ \bullet) \equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b$$
$$\mathbb{RC}(X,\ \bullet) \equiv \forall a, b, c \in X,\ a \bullet c = b \bullet c \Rightarrow a = b$$

**Left and Right Constant**

$$\mathbb{LK}(X,\ \bullet) \equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b$$
$$\mathbb{RK}(X,\ \bullet) \equiv \forall a, b, c \in X,\ a \bullet c = b \bullet c$$



## Why bisemigroups?

But wait! How could any semiring satisfy either of these properties?

$$\mathbb{LC}(X,\ \bullet) \equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b$$
$$\mathbb{LK}(X,\ \bullet) \equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b$$

- For $\mathbb{LC}$, note that we always have $\overline{0} \otimes a = \overline{0} \otimes b$, so $\mathbb{LC}$ could only hold when $S = \{\overline{0}\}$.
- For $\mathbb{LK}$, let $a = \overline{1}$ and $b = \overline{0}$ and $\mathbb{LK}$ leads to the conclusion that every $c$ is equal to $\overline{0}$ (again!).

Normally we will add a zero and/or a one as the last step(s) of constructing a semiring. Alternatively, we might want to complicate our properties so that things work for semirings. A design trade-off!

## Proof of $\Leftarrow$ for $\mathbb{LD}$

Assume

$$\begin{array}{ll} (1) & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \\ (2) & \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\ (3) & \mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T) \\ (4) & \mathbb{IP}(S,\ \oplus_S). \end{array}$$

Let $\oplus \equiv \oplus_S \vec{\times} \oplus_T$ and $\otimes \equiv \otimes_S \times \otimes_T$. Suppose

$$(s_1, t_1),\ (s_2, t_2),\ (s_3, t_3) \in S \times T.$$

We want to show that

$$\begin{array}{rcl} \text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\ & = & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\ & \equiv & \text{rhs} \end{array}$$



## Proof of $\Leftarrow$ for $\mathbb{LD}$

We have

$$\begin{array}{rcl} \text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\ & = & (s_1,\ t_1) \otimes (s_2 \oplus_S s_3,\ t_{\text{lhs}}) \\ & = & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_1 \otimes_T t_{\text{lhs}}) \end{array}$$

$$\begin{array}{rcl} \text{rhs} & \equiv & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\ & = & (s_1 \otimes_S s_2,\ t_1 \otimes_T t_2) \oplus (s_1 \otimes_S s_3,\ t_1 \otimes_T t_3) \\ & = & ((s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3),\ t_{\text{rhs}}) \\ & =_{(1)} & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_{\text{rhs}}) \end{array}$$

where $t_{\text{lhs}}$ and $t_{\text{rhs}}$ are determined by the appropriate case in the definition of $\oplus$. Finally, note that

$$\text{lhs} = \text{rhs} \Leftrightarrow t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}.$$

## Proof by cases on $s_2 \oplus_S s_3$

Case 1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus_T t_3$ and

$$t_1 \otimes_T t_{\text{lhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) =_{(2)} (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3).$$

Since $s_2 = s_3$ we have $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and

$$s_1 \otimes_S s_2 =_{(4)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3) =_{(4)} s_1 \otimes_S s_3.$$

Therefore,

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus (t_1 \otimes_T t_3) = t_1 \otimes_T t_{\text{lhs}}.$$

Case 2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_{\text{lhs}} = t_2$ and

$$t_1 \otimes_T t_{\text{lhs}} = t_1 \otimes_T t_2.$$

Since $s_2 = s_2 \oplus_S s_3$ we have

$$s_1 \otimes_S s_2 = s_1 \otimes_S (s_2 \oplus_S s_3) =_{(1)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3).$$



Case 2.1 $s_1 \otimes_S s_2 \neq s_1 \otimes_S s_3$. Then $t_{\text{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}$.
Case 2.2 $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$. Then

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) =_{(2)} t_1 \otimes_T (t_2 \oplus_T t_3)$$

We need to consider two subcases.
Case 2.2.1: Assume $\mathbb{LC}(S, \otimes_S)$. But $s_1 \otimes_S s_2 = s_1 \otimes_S s_3 \Rightarrow s_2 = s_3$, which is a contradiction.
Case 2.2.2 : Assume $\mathbb{LK}(T, \otimes_T)$. In this case we know

$$\forall a, b \in X,\ t_1 \otimes_T a = t_1 \otimes_T b.$$

Letting $a = t_2 \oplus_T t_3$ and $b = t_2$ we have

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}.$$

Case 3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to Case 2.

## Other direction, $\Rightarrow$

Prove this:

$$\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \vee \neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \vee (\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$$
$$\Rightarrow\ \neg\mathbb{LD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T)).$$

Case 1: $\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S)$. That is

$$\exists a, b, c \in S,\ a \otimes_S (b \oplus_S c) \neq (a \otimes_S b) \oplus_S (a \otimes_S c).$$

Pick any $t \in T$. Then for some $t_1, t_2, t_3 \in T$ we have

$$\begin{array}{ll} & (a,\ t) \otimes ((b,\ t) \oplus (c,\ t)) \\ = & (a,\ t) \otimes (b \oplus_S c,\ t_1) \\ = & (a, \otimes_S (b \oplus_S c),\ t_2) \\ \neq & ((a \otimes_S b) \oplus_S (a \otimes_S c),\ t_3) \\ = & (a \otimes_S b,\ t \otimes_T t) \oplus (a \otimes_S c,\ t \otimes_T t) \\ = & ((a,\ t) \otimes (b,\ t)) \oplus ((a,\ t) \otimes (c,\ t)) \end{array}$$

Case 2: $\neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$. Similar.



Case 3: $(\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$. That is

$$\exists a, b, c \in S,\ c \otimes_S a = c \otimes_S b \wedge a \neq b$$

and

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

Since $\oplus_S$ is selective and $a \neq b$, we have $a = a \oplus_S b$ or $b = a \oplus_S b$.
Assume without loss of generality that $a = a \oplus_S b \neq b$.
Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{lll} \text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\ & = & (c,\ t_1) \otimes (a,\ t_2) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \end{array}$$

$$\begin{array}{lll} \text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\ & = & (c \otimes_S a,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \end{array}$$

Our job now is to select $t_1, t_2, t_3$ so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) \equiv t_{\text{rhs}}.$$

We don't have very much to work with! Only

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

In addition, we can assume $\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$ (otherwise, use Case 2!), so

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3).$$



## We need to select $t_1, t_2, t_3$ so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\text{rhs}}.$$

Case 3.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.2: $z \otimes_T y = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T y = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.3: $z \otimes_T x \neq z \otimes_T (x \oplus_T y) \neq z \otimes_T y$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

□

## We have to start somewhere!

| $S$ | $\oplus$ | $\otimes$ | $\overline{0}$ | $\overline{1}$ | $\mathbb{LD}$ | $\mathbb{RD}$ | $\mathbb{ZA}$ | $\mathbb{OA}$ | $\mathbb{LAB}$ | $\mathbb{RAB}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{N}$ | min | $+$ | | 0 | $\star$ | $\star$ | | $\star$ | $\star$ | $\star$ |
| $\mathbb{N}$ | max | $+$ | 0 | 0 | $\star$ | $\star$ | | | $\star$ | $\star$ |
| $\mathbb{N}$ | max | min | 0 | | $\star$ | $\star$ | $\star$ | | $\star$ | $\star$ |
| $\mathbb{N}$ | min | max | | 0 | $\star$ | $\star$ | | $\star$ | $\star$ | $\star$ |



## Widest shortest paths

$$\mathrm{wsp} \equiv \mathrm{AddZero}(\infty_2,\ (\mathbb{N},\ \mathsf{min},\ +)\ \vec{\times}\ \mathrm{AddOne}(\infty_1,\ (\mathbb{N},\ \mathsf{max},\ \mathsf{min})))$$

$$= ((\mathbb{N} \times (\mathbb{N} \uplus \{\infty_1\})) \uplus \{\infty_2\},\ \oplus,\ \otimes,\ \mathrm{inr}(\infty_2),\ \mathrm{inl}(0,\ \mathrm{inr}(\infty_1)))$$

where

$$\oplus = (\mathsf{min}\ \vec{\times}\ \mathsf{max}^{\mathrm{an}}_{\infty_1})^{\mathrm{id}}_{\infty_2}$$

$$\otimes = (+ \times \mathsf{min}^{\mathrm{id}}_{\infty_1})^{\mathrm{an}}_{\infty_2}$$

## Example

$$
\begin{array}{ll}
 & \mathrm{inl}(3, \mathrm{inl}(10)) \otimes (\mathrm{inl}(17, \mathrm{inl}(21)) \oplus \mathrm{inl}(11, \mathrm{inl}(4))) \\
= & \mathrm{inl}(3, \mathrm{inl}(10)) \otimes \mathrm{inl}(11, \mathrm{inl}(4)) \\
= & \mathrm{inl}(14, \mathrm{inl}(4))
\end{array}
$$

$$
\begin{array}{ll}
 & (\mathrm{inl}(3, \mathrm{inl}(10)) \otimes \mathrm{inl}(17, \mathrm{inl}(21))) \oplus (\mathrm{inl}(3, \mathrm{inl}(10)) \otimes \mathrm{inl}(11, \mathrm{inl}(4))) \\
= & \mathrm{inl}(20, \mathrm{inl}(10)) \oplus \mathrm{inl}(14, \mathrm{inl}(4)) \\
= & \mathrm{inl}(14, \mathrm{inl}(4))
\end{array}
$$

But is $\mathrm{wsp}$ a semiring?



## Turn the cranks!

### Turning the crank for $\mathbb{LD}$:

$$
\begin{array}{ll}
 & \mathbb{LD}(\mathrm{AddZero}(\infty_2, (\mathbb{N}, \text{min}, +) \vec{\times} \mathrm{AddOne}(\infty_1, (\mathbb{N}, \text{max}, \text{min})))) \\
\Leftrightarrow & \mathbb{LD}((\mathbb{N}, \text{min}, +) \vec{\times} \mathrm{AddOne}(\infty_1, (\mathbb{N}, \text{max}, \text{min}))) \\
\Leftrightarrow & \mathbb{LD}(\mathbb{N}, \text{min}, +) \wedge \mathbb{LD}(\mathrm{AddOne}(\infty_1, (\mathbb{N}, \text{max}, \text{min}))) \\
 & \wedge (\mathbb{LC}(\mathbb{N}, +) \vee \mathbb{LK}(\mathrm{AddID}(\infty_1, (\mathbb{N}, \text{min})))) \\
\Leftrightarrow & \mathbb{TRUE} \wedge (\mathbb{LD}(\mathbb{N}, \text{max}, \text{min}) \wedge \mathbb{RAB}(\mathbb{N}, \text{max}, \text{min}) \wedge \mathbb{IP}(\mathbb{N}, \text{max})) \\
 & \wedge (\mathbb{TRUE} \vee \mathbb{LK}(\mathrm{AddID}(\infty_1, (\mathbb{N}, \text{min})))) \\
\Leftrightarrow & \mathbb{TRUE} \wedge (\mathbb{TRUE} \wedge \mathbb{TRUE} \wedge \mathbb{TRUE}) \\
 & \wedge (\mathbb{TRUE} \vee \mathbb{LK}(\mathrm{AddID}(\infty_1, (\mathbb{N}, \text{min})))) \\
\Leftrightarrow & \mathbb{TRUE}
\end{array}
$$