L11: Algebraic Path Problems with applications to Internet Routing Lecture 4

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Lecture 4

Let us pause to look at two interesting semirings ...

- Mini-max
- Martelli's semiring for computing sets of minimal cut sets

A Minimax Semiring

$$\text{minimax} \equiv (\mathbb{N}^{\infty}, \min, \max, \infty, 0)$$

$$17 \min \infty = 17$$

$$17 \max \infty = \infty$$

How can we interpret this?

$$\mathbf{A}^*(i, j) = \min_{p \in \pi(i, j)} \max_{(u,v) \in p} \mathbf{A}(u, v),$$

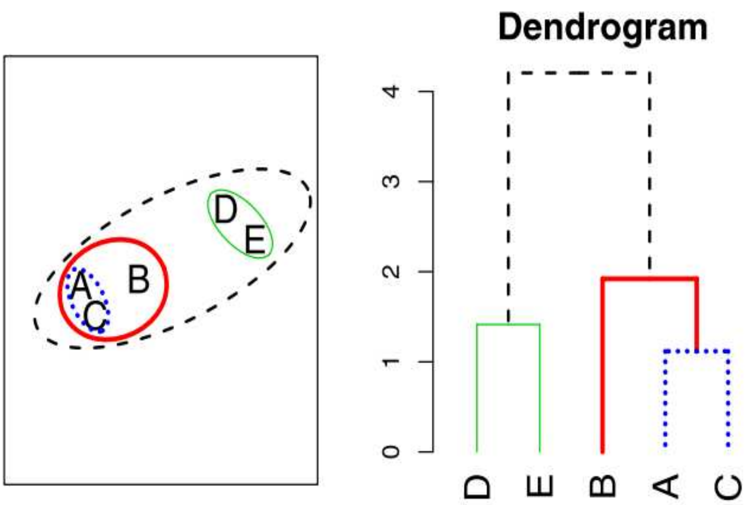
One possible interpretation of Minimax

- Given an adjacency matrix $\mathbf{A}$ over minimax,
- suppose that $\mathbf{A}(i, j) = 0 \Leftrightarrow i = j$,
- suppose that $\mathbf{A}$ is symmetric ($\mathbf{A}(i, j) = \mathbf{A}(j, i)$),
- interpret $\mathbf{A}(i, j)$ as measured dissimilarity of i and j,
- interpret $\mathbf{A}^*(i, j)$ as inferred dissimilarity of i and j,

Many uses

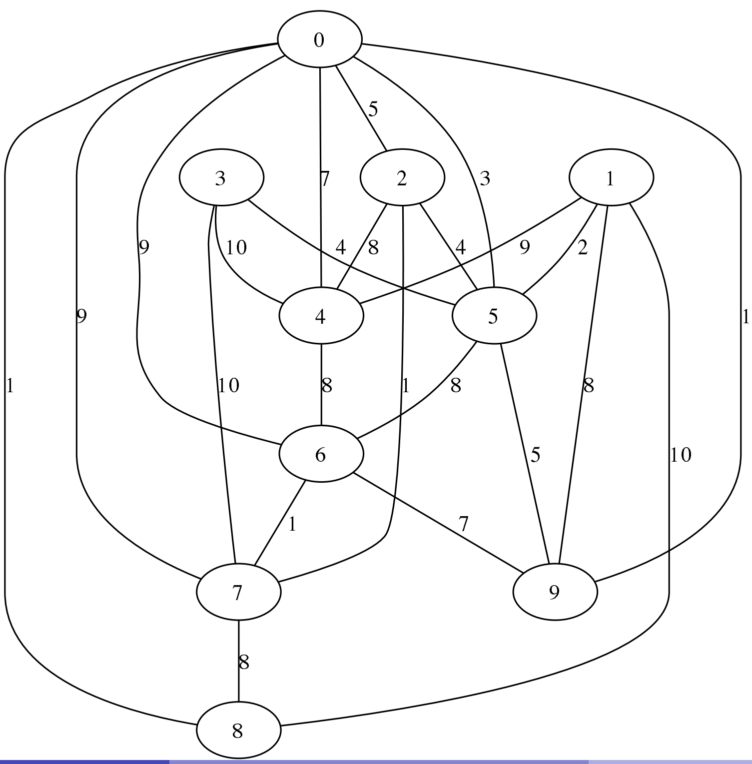
- Hierarchical clustering of large data sets
- Classification in Machine Learning
- Computational phylogenetics
- ...

Dendrograms

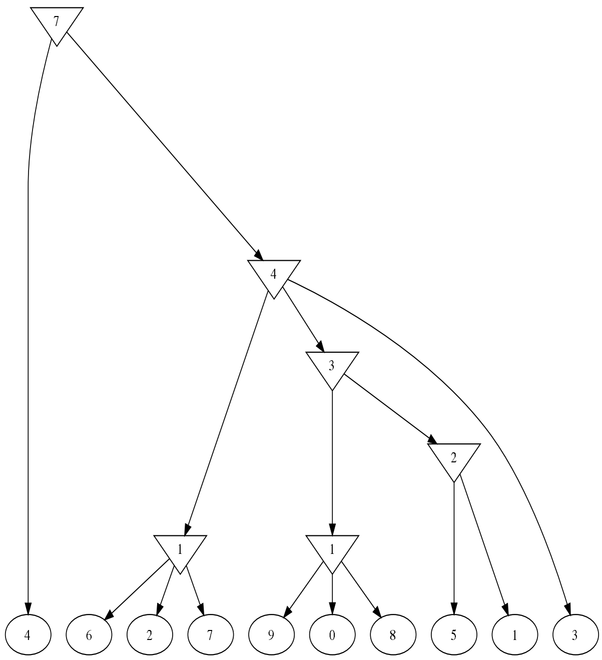


from **Hierarchical Clustering With Prototypes via Minimax Linkage**, Bien and Tibshirani, 2011.

A minimax graph



The solution $\mathbf{A}^*$ drawn as a dendrogram



Hierarchical clustering? Why?

Suppose $(Y, \leqslant, +)$ is a totally ordered with least element 0.

Metric

A <u>metric</u> for set X over $(Y, \leqslant, +)$ is a function $d \in X \times X \to Y$ such that

- $\forall x, y \in X,\ d(x, y) = 0 \Leftrightarrow x = y$
- $\forall x, y \in X,\ d(x, y) = d(y, x)$
- $\forall x, y, z \in X,\ d(x, y) \leqslant d(x, z) + d(z, y)$

Ultrametric

An <u>ultrametric</u> for set X over $(Y, \leqslant)$ is a function $d \in X \times X \to Y$ such that

- $\forall x \in X,\ d(x, x) = 0$
- $\forall x, y \in X,\ d(x, y) = d(y, x)$
- $\forall x, y, z \in X,\ d(x, y) \leqslant d(x, z) \max d(z, y)$

Fun Facts

Fact 5

If $\mathbf{A}$ is an $n \times n$ symmetric minimax adjacency matrix, then $\mathbf{A}^*$ is a finite ultrametric for $\{0, 1, \ldots, n-1\}$ over $(\mathbb{N}^\infty, \leqslant))$.

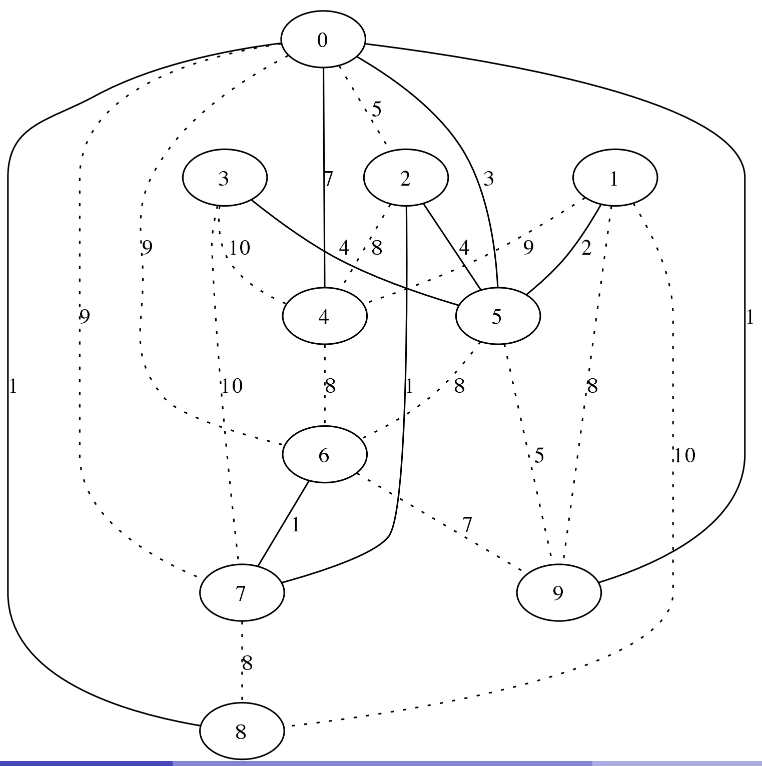
Fact 6

Suppose each arc weight is unique. Then the set of arcs

$$\{(i, j) \in E \mid \mathbf{A}(i, j) = \mathbf{A}^*(i, j)\}$$

is a minimum spanning tree.

A spanning tree derived from $\mathbf{A}$ and $\mathbf{A}^*$



Cut Sets

Let $G = (V,\ E)$ be a directed graph.

- A cut set $C \subseteq E$ for nodes i and j is a set of arcs such there is no path from i to j in the graph $(V,\ E - C)$.
- C is minimal if no proper subset of C is an arc cut set.

Martelli's Semiring

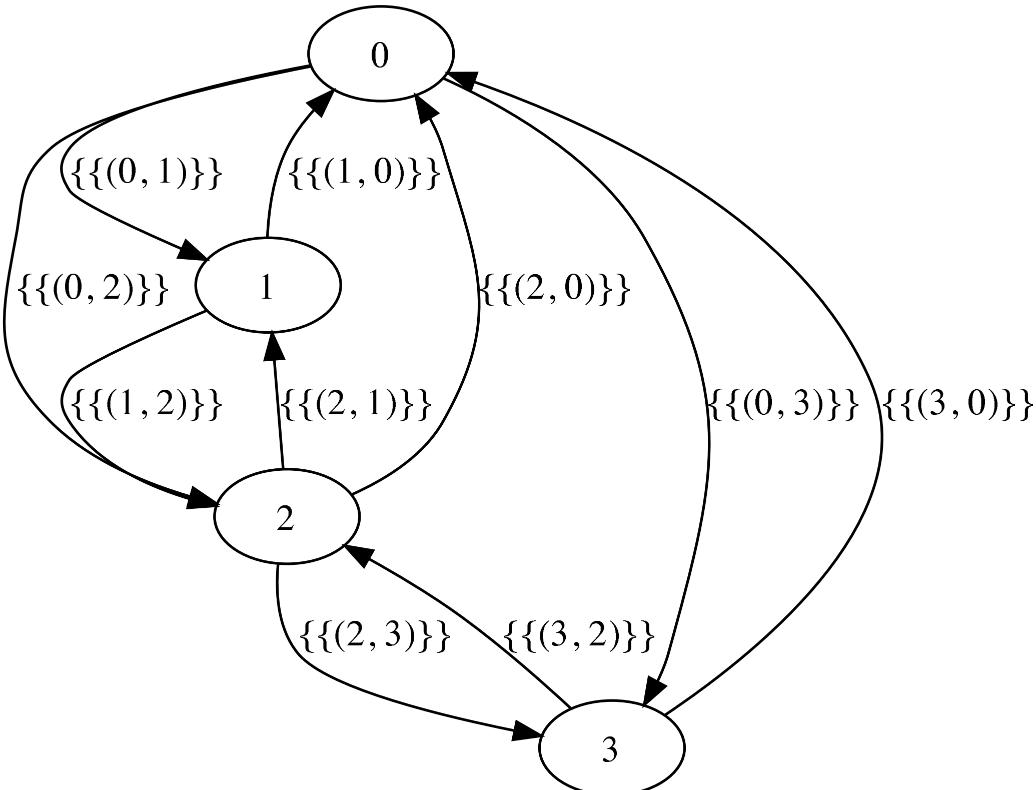
Let $G = (V,\ E)$ be a directed graph.

$$
\begin{array}{rcl}
\mathrm{M} & \equiv & (S,\ \oplus,\ \otimes,\ \overline{0},\ \overline{1}) \\
S & \equiv & \{X \in 2^{2^E} \mid \forall U, V \in X, U \subseteq V \implies U = V\} \\
X \oplus Y & \equiv & \text{remove all supersets from } \{U \cup V \mid U \in X,\ V \in Y\} \\
X \otimes Y & \equiv & \text{remove all supersets from } X \cup Y \\
\overline{0} & \equiv & \{\{\}\} \\
\overline{1} & \equiv & \{\}
\end{array}
$$

What does it do?

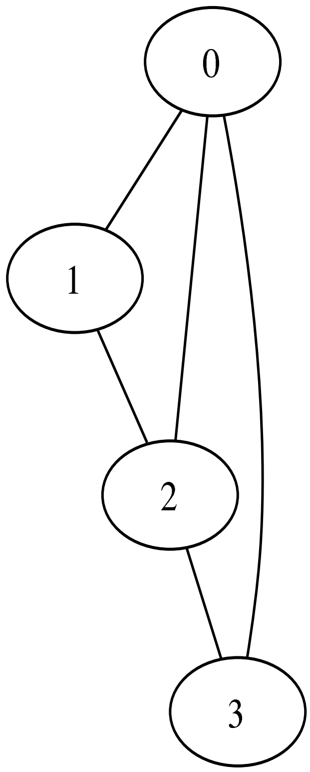
- If every arc $(i,\ j)$ is has weight $\mathbf{A}(i,\ j) = \{\{(i,\ j)\}\}$, then $\mathbf{A}^*(i,\ j)$ is the set of all minimal arc cut sets for i and j.

A



Part of **A***

$$
\begin{aligned}
\mathbf{A}^*(0,\ 1) &= \{\{(0,1),(2,1)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,1),(0,2),(3,2)\}\}\\
\mathbf{A}^*(0,\ 2) &= \{\{(0,2),(1,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,2),(0,3),(1,2)\}\}\\
\mathbf{A}^*(2,\ 0) &= \{\{(2,0),(2,1),(3,0)\},\\
&\quad \{(1,0),(2,0),(3,0)\},\\
&\quad \{(1,0),(2,0),(2,3)\},\\
&\{(2,0),(2,1),(2,3)\}\}\\
\mathbf{A}^*(2,\ 3) &= \{\{(2,0),(2,1),(2,3)\},\\
&\quad \{(0,3),(2,3)\},\\
&\quad \{(1,0),(2,0),(2,3)\}\}
\end{aligned}
$$



Homework 2 (due 27 October)

- Prove Fun Fact 6.
- Prove that distributivity holds for Martelli's semiring.