L11: Algebraic Path Problems with applications to Internet Routing Lecture 3

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Bi-semigroups and Pre-Semirings

$(S, \oplus, \otimes)$ is a bi-semigroup when

- $(S, \oplus)$ is a semigroup
- $(S, \otimes)$ is a semigroup

$(S, \oplus, \otimes)$ is a pre-semiring when

- $(S, \oplus, \otimes)$ is a bi-semigroup
- $\oplus$ is commutative

and left- and right-distributivity hold,

$$
\begin{array}{lcrcl}
\mathbb{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\
\mathbb{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c)
\end{array}
$$

Semirings

$(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring when

- $(S, \oplus, \otimes)$ is a pre-semiring
- $(S, \oplus, \overline{0})$ is a (commutative) monoid
- $(S, \otimes, \overline{1})$ is a monoid
- $\overline{0}$ is an annihilator for $\otimes$

Examples

Pre-semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
min_plus	$\mathbb{N}$	min	$+$		0
max_min	$\mathbb{N}$	max	min	0	

Semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0
bw	$\mathbb{N}^{\infty}$	max	min	0	∞

Note the sloppiness — the symbols $+$, max, and min in the two tables represent different functions....

How about $(\max, +)$?

Pre-semiring

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
max_plus	$\mathbb{N}$	max	$+$	0	0

- What about "$\overline{0}$ is an annihilator for $\otimes$"? No!

Fix that ...

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
$\text{max\_plus}^{-\infty}$	$\mathbb{N} \uplus \{-\infty\}$	max	$+$	$-\infty$	0

Stability

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring

$a \in S$, define powers a^k

$$\begin{aligned} a^0 &= \overline{1} \\ a^{k+1} &= a \otimes a^k \end{aligned}$$

Closure, a^*

$$\begin{aligned} a^{(k)} &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\ a^* &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots \end{aligned}$$

Definition (q stability)

If there exists a q such that $a^{(q)} = a^{(q+1)}$, then a is q-stable. By induction: $\forall t, 0 \leqslant t, a^{(q+t)} = a^{(q)}$. Therefore, $a^* = a^{(q)}$.

Matrix Semirings

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- Define the semiring of $n \times n$-matrices over S : $(\mathbb{M}_n(S), \oplus, \otimes, \mathbf{J}, \mathbf{I})$

$\oplus$ and $\otimes$

$$\begin{aligned} (\mathbf{A} \oplus \mathbf{B})(i, j) &= \mathbf{A}(i, j) \oplus \mathbf{B}(i, j) \\ (\mathbf{A} \otimes \mathbf{B})(i, j) &= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j) \end{aligned}$$

$\mathbf{J}$ and $\mathbf{I}$

$$\begin{aligned} \mathbf{J}(i, j) &= \overline{0} \\ \mathbf{I}(i, j) &= \begin{cases} \overline{1} & (\text{if } i = j) \\ \overline{0} & (\text{otherwise}) \end{cases} \end{aligned}$$

$\mathbb{M}_n(S)$ is a semiring!

For example, here is left distribution

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

$$\begin{aligned} & (\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}))(i, j) \\ = & \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes (\mathbf{B} \oplus \mathbf{C})(q, j) \\ = & \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes (\mathbf{B}(q, j) \oplus \mathbf{C}(q, j)) \\ = & \bigoplus_{1 \leqslant q \leqslant n} (\mathbf{A}(i, q) \otimes \mathbf{B}(q, j)) \oplus (\mathbf{A}(i, q) \otimes \mathbf{C}(q, j)) \\ = & (\bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)) \oplus (\bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes \mathbf{C}(q, j)) \\ = & ((\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}))(i, j) \end{aligned}$$

Note : we only needed left-distributivity on S.

Matrix encoding path problems

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- $G = (V, E)$ a directed graph
- $w \in E \to S$ a weight function

Path weight

The weight of a path $p = i_1, i_2, i_3, \cdots, i_k$ is

$$w(p) = w(i_1, i_2) \otimes w(i_2, i_3) \otimes \cdots \otimes w(i_{k-1}, i_k).$$

The empty path is given the weight $\overline{1}$.

Adjacency matrix **A**

$$\mathbf{A}(i, j) = \begin{cases} w(i, j) & \text{if } (i, j) \in E, \\ \overline{0} & \text{otherwise} \end{cases}$$

The general problem of finding globally optimal path weights

Given an adjacency matrix $\mathbf{A}$, find $\mathbf{A}^*$ such that for all $i, j \in V$

$$\mathbf{A}^*(i, j) = \bigoplus_{p \in \pi(i, j)} w(p)$$

where $\pi(i, j)$ represents the set of all paths from i to j.

How can we solve this problem?

Matrix methods

Matrix powers, $\mathbf{A}^k$

$$\mathbf{A}^0 = \mathbf{I}$$

$$\mathbf{A}^{k+1} = \mathbf{A} \otimes \mathbf{A}^k$$

Closure, $\mathbf{A}^*$

$$\mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots$$

Note: $\mathbf{A}^*$ might not exist. Why?

Matrix methods can compute optimal path weights

- Let $\pi(i, j)$ be the set of paths from i to j.
- Let $\pi^k(i, j)$ be the set of paths from i to j with exactly k arcs.
- Let $\pi^{(k)}(i, j)$ be the set of paths from i to j with at most k arcs.

Theorem

$$\begin{array}{lrcl}
(1) & \mathbf{A}^k(i,\ j) & = & \bigoplus_{p \in \pi^k(i,\ j)} w(p) \\
(2) & \mathbf{A}^{(k)}(i,\ j) & = & \bigoplus_{p \in \pi^{(k)}(i,\ j)} w(p) \\
(3) & \mathbf{A}^*(i,\ j) & = & \bigoplus_{p \in \pi(i,\ j)} w(p)
\end{array}$$

Warning again: for some semirings the expression $\mathbf{A}^*(i,\ j)$ might not be well-defeind. Why?

Proof of (1)

By induction on k. Base Case: $k = 0$.

$$\pi^0(i,\ i) = \{\epsilon\},$$

so $\mathbf{A}^0(i\ , i) = \mathbf{I}(i,\ i) = \overline{1} = w(\epsilon)$.

And $i \neq j$ implies $\pi^0(i, j) = \{\}$. By convention

$$\bigoplus_{p \in \{\}} w(p) = \overline{0} = \mathbf{I}(i,\ j).$$

Proof of (1)

Induction step.

$$\begin{aligned}
\mathbf{A}^{k+1}(i, j) &= (\mathbf{A} \otimes \mathbf{A}^k)(i,\ j) \\
&= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i,\ q) \otimes \mathbf{A}^k(q,\ j) \\
&= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i,\ q) \otimes \Big(\bigoplus_{p \in \pi^k(q,\ j)} w(p)\Big) \\
&= \bigoplus_{1 \leqslant q \leqslant n} \bigoplus_{p \in \pi^k(q,\ j)} \mathbf{A}(i,\ q) \otimes w(p) \\
&= \bigoplus_{(i,\ q) \in E} \bigoplus_{p \in \pi^k(q, j)} w(i,\ q) \otimes w(p) \\
&= \bigoplus_{p \in \pi^{k+1}(i,\ j)} w(p)
\end{aligned}$$

Fun Facts

Fact 3

If $\overline{1}$ is an annihiltor for $\oplus$, then every $a \in S$ is 0-stable!

Fact 4

If S is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

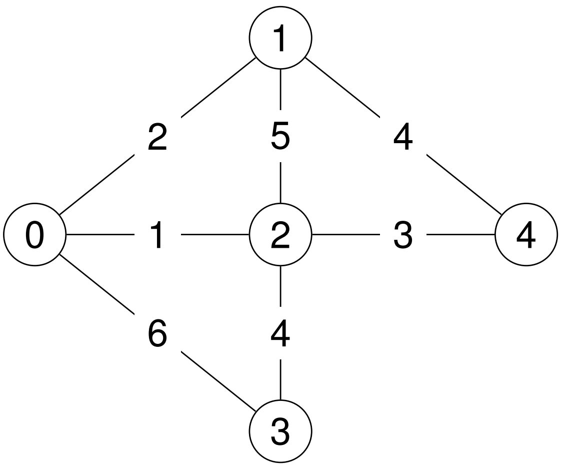
$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

Why? Because we can ignore paths with loops.

$$(a \otimes c \otimes b) \oplus (a \otimes b) = a \otimes (\overline{1} \oplus c) \otimes b = a \otimes \overline{1} \otimes b = a \otimes b$$

Think of c as the weight of a loop in a path with weight $a \otimes b$.

Shortest paths example, $(\mathbb{N}^{\infty}, \ \min, \ +)$



The adjacency matrix

$$\mathbf{A} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{bmatrix} \end{array}$$

Note that the longest shortest path is $(1, \ 0, \ 2, \ 3)$ of length 3 and weight 7.

$(\min, +)$ example

Our theorem tells us that $\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{A}^{(4)}$

$$\mathbf{A}^* = \mathbf{A}^{(4)} = \mathbf{I} \ \min \ \mathbf{A} \ \min \ \mathbf{A}^2 \ \min \ \mathbf{A}^3 \ \min \ \mathbf{A}^4 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

$(\min, +)$ example

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \infty & \underline{2} & \underline{1} & 6 & \infty \\ \underline{2} & \infty & 5 & \infty & \underline{4} \\ \underline{1} & 5 & \infty & \underline{4} & \underline{3} \\ 6 & \infty & \underline{4} & \infty & \infty \\ \infty & \underline{4} & \underline{3} & \infty & \infty \end{bmatrix} \end{array} \qquad \mathbf{A}^3 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 8 & 4 & 3 & 8 & 10 \\ 4 & 8 & 7 & \underline{7} & 6 \\ 3 & 7 & 8 & 6 & 5 \\ 8 & \underline{7} & 6 & 11 & 10 \\ 10 & 6 & 5 & 10 & 12 \end{bmatrix} \end{array}$$

$$\mathbf{A}^2 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2 & 6 & 7 & \underline{5} & \underline{4} \\ 6 & 4 & \underline{3} & 8 & 8 \\ 7 & \underline{3} & 2 & 7 & 9 \\ \underline{5} & 8 & 7 & 8 & \underline{7} \\ \underline{4} & 8 & 9 & \underline{7} & 6 \end{bmatrix} \end{array} \qquad \mathbf{A}^4 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 4 & 8 & 9 & 7 & 6 \\ 8 & 6 & 5 & 10 & 10 \\ 9 & 5 & 4 & 9 & 11 \\ 7 & 10 & 9 & 10 & 9 \\ 6 & 10 & 11 & 9 & 8 \end{bmatrix} \end{array}$$

First appearance of final value is in red and underlined. Remember: we are looking at all paths of a given length, even those with cycles!

A “better” way — our basic algorithm

$$
\begin{array}{rcl}
\mathbf{A}^{\langle 0 \rangle} & = & \mathbf{I} \\
\mathbf{A}^{\langle k+1 \rangle} & = & \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I}
\end{array}
$$

Lemma

$$\mathbf{A}^{\langle k \rangle} = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

back to $(\min, +)$ example

$$
\mathbf{A}^{\langle 1 \rangle} = \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{ccccc}
0 & 2 & 1 & 6 & \infty \\
2 & 0 & 5 & \infty & 4 \\
1 & 5 & 0 & 4 & 3 \\
6 & \infty & 4 & 0 & \infty \\
\infty & 4 & 3 & \infty & 0
\end{array}\right]
\end{array}
\quad
\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{ccccc}
0 & 2 & 1 & 5 & 4 \\
2 & 0 & 3 & 7 & 4 \\
1 & 3 & 0 & 4 & 3 \\
5 & 7 & 4 & 0 & 7 \\
4 & 4 & 3 & 7 & 0
\end{array}\right]
\end{array}
$$

$$
\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{ccccc}
0 & 2 & 1 & 5 & 4 \\
2 & 0 & 3 & 8 & 4 \\
1 & 3 & 0 & 4 & 3 \\
5 & 8 & 4 & 0 & 7 \\
4 & 4 & 3 & 7 & 0
\end{array}\right]
\end{array}
$$

A note on $\mathbf{A}$ vs. $\mathbf{A} \oplus \mathbf{I}$

Lemma

If $\oplus$ is idempotent, then

$$(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}.$$

Proof. Base case: When $k = 0$ both expressions are $\mathbf{I}$.
Assume $(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}$. Then

$$\begin{array}{lcl}
(\mathbf{A} \oplus \mathbf{I})^{k+1} & = & (\mathbf{A} \oplus \mathbf{I})(\mathbf{A} \oplus \mathbf{I})^k \\
& = & (\mathbf{A} \oplus \mathbf{I})\mathbf{A}^{(k)} \\
& = & \mathbf{A}\mathbf{A}^{(k)} \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A}(\mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^k) \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A}^{(k+1)}
\end{array}$$

Solving (some) equations

Theorem 6.1

If $\mathbf{A}$ is q-stable, then $\mathbf{A}^*$ solves the equations

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{I}$$

and

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{I}.$$

For example, to show $\mathbf{L} = \mathbf{A}^*$ solves the first equation:

$$\begin{array}{lcl}
\mathbf{A}^* & = & \mathbf{A}^{(q)} \\
& = & \mathbf{A}^{(q+1)} \\
& = & \mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I} \\
& = & \mathbf{A}(\mathbf{A}^q \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I}) \oplus \mathbf{I} \\
& = & \mathbf{A}\mathbf{A}^{(q)} \oplus \mathbf{I} \\
& = & \mathbf{A}\mathbf{A}^* \oplus \mathbf{I}
\end{array}$$

Note that if we replace the assumption "$\mathbf{A}$ is q-stable" with "$\mathbf{A}^*$ exists," then we require that $\otimes$ distributes over infinite sums.

A more general result

Theorem Left-Right

If $\mathbf{A}$ is q-stable, then $\mathbf{L} = \mathbf{A}^*\mathbf{B}$ solves the equation

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{B}$$

and $\mathbf{R} = \mathbf{B}\mathbf{A}^*$ solves

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{B}.$$

For the first equation:

$$\begin{array}{rcl}
\mathbf{A}^*\mathbf{B} & = & \mathbf{A}^{(q)}\mathbf{B} \\
& = & \mathbf{A}^{(q+1)}\mathbf{B} \\
& = & (\mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \\
& = & (\mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A})\mathbf{B} \oplus \mathbf{B} \\
& = & \mathbf{A}(\mathbf{A}^q \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \oplus \mathbf{B} \\
& = & \mathbf{A}(\mathbf{A}^{(q)}\mathbf{B}) \oplus \mathbf{B} \\
& = & \mathbf{A}(\mathbf{A}^*\mathbf{B}) \oplus \mathbf{B}
\end{array}$$

The “best” solution

Suppose $\mathbf{Y}$ is a matrix such that

$$\mathbf{Y} = \mathbf{A}\mathbf{Y} \oplus \mathbf{I}$$

$$\begin{array}{rcl}
\mathbf{Y} & = & \mathbf{A}\mathbf{Y} \oplus \mathbf{I} \\
& = & \mathbf{A}^1\mathbf{Y} \oplus \mathbf{A}^{(0)} \\
& = & \mathbf{A}((\mathbf{A}\mathbf{Y} \oplus \mathbf{I})) \oplus \mathbf{I} \\
& = & \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A} \oplus \mathbf{I} \\
& = & \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A}^{(1)} \\
\vdots & & \vdots \quad \vdots \\
& = & \mathbf{A}^{k+1}\mathbf{Y} \oplus \mathbf{A}^{(k)}
\end{array}$$

If $\mathbf{A}$ is q-stable and $q < k$, then

$$\mathbf{Y} = \mathbf{A}^k\mathbf{Y} \oplus \mathbf{A}^*$$

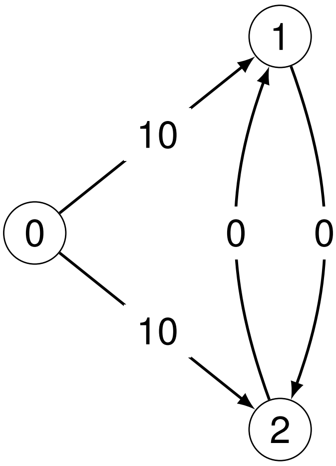
$$\mathbf{Y} \trianglelefteq^L_{\oplus} \mathbf{A}^*$$

and if $\oplus$ is idempotent, then

$$\mathbf{Y} \leq^L_{\oplus} \mathbf{A}^*$$

So $\mathbf{A}^*$ is the largest solution. What does this mean in terms of the sp semiring?

Example with zero weighted cycles using sp semiring



$$\mathbf{A} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \infty & 10 & 10 \\ 1 & \infty & \infty & 0 \\ 2 & \infty & 0 & \infty \end{array}$$

$\mathbf{A}^*$ $(= \mathbf{A} \oplus \mathbf{I}$ in this case) solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

But so does this (dishonest) matrix!

$$\mathbf{F} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 9 & 9 \\ 1 & \infty & 0 & 0 \\ 2 & \infty & 0 & 0 \end{array}$$

For example :

$$\begin{aligned} & (\mathbf{F}\mathbf{A} \oplus \mathbf{I})(0,1) \\ = & \min_{q \in \{0,1,2\}} \mathbf{F}(0,q) + \mathbf{A}(q,1) \\ = & \min(0+10, 9+\infty, 9+0) \\ = & 9 \\ = & \mathbf{F}(0,1) \end{aligned}$$