L11: Algebraic Path Problems with applications to Internet Routing Lecture 12

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Michaelmas Term, 2017

Lecture 15

k shortest paths

- We need at least one example of an interesting, non-idempotent, semiring.
- Recommended reading: Semiring frameworks and algorithms for shortest-distance problems, Mehryar Mohri, Journal of Automata, Languages and Combinatorics, v7, number 2, 2002

k shortest paths

The $\mathcal{T}_k$ semiring

$$\mathcal{T}_k \equiv (\mathbb{T}_k,\ \oplus_k,\ \otimes_k,\ \overline{0}_k,\ \overline{1}_k)$$

where

$$\begin{aligned}
(a_0,\ \ldots,\ a_k) \oplus_k (b_0,\ \ldots,\ b_k) &\equiv \min_k(a_0, \ldots,\ a_k,\ b_0,\ \ldots,\ b_k) \\
\overline{0}_k &\equiv (\infty,\ \infty,\ \cdots,\ \infty) \\
(a_0,\ \ldots,\ a_k) \otimes_k (b_0,\ \ldots,\ b_k) &\equiv \min_k(a_0 + b_0,\ a_0 + b_1,\ \ldots,\ a_k + b_k) \\
\overline{1}_k &\equiv (0,\ \infty,\ \cdots,\ \infty)
\end{aligned}$$

$\mathcal{T}_k$ is $(k-1)$-stable.

Examples ($\oplus_2$). Note that $\mathcal{T}_k$ is not idempotent for $k > 1$.

$$\begin{aligned}
(5,\ 8) \oplus_2 (3,\ 6) &= \min_2(5,\ 8,\ 3,\ 6) \\
&= (3,\ 5)
\end{aligned}$$

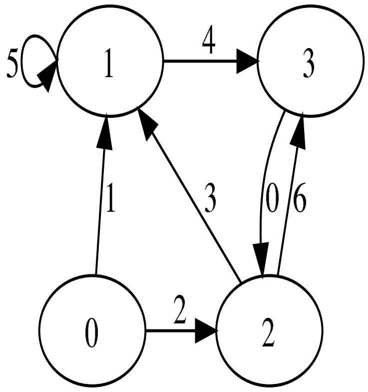
$$\begin{aligned}
(1,\ 20) \oplus_2 (1,\ 20) &= \min_2(1,\ 20,\ 1,\ 20) \\
&= (1,\ 1)
\end{aligned}$$

Examples ($\otimes_2$)

$$\begin{aligned}
(5,\ 8) \otimes_2 (3,\ 6) &= \min_2(5+3,\ 5+6,\ 8+3,\ 8+6) \\
&= \min_2(8,\ 11,\ 11,\ 14) \\
&= (8,\ 11)
\end{aligned}$$

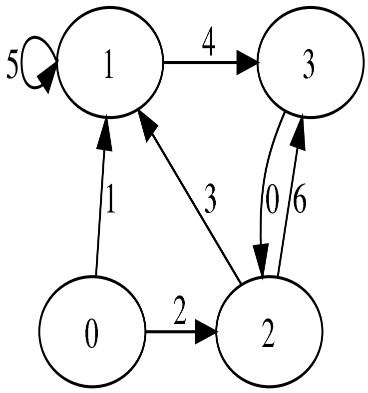
$$\begin{aligned}
(5,\ 8) \otimes_2 \overline{0}_2 &= \min_2(5+\infty,\ 5+\infty,\ 8+\infty,\ 8+\infty) \\
&= \min_2(\infty,\ \infty,\ \infty,\ \infty) \\
&= (\infty,\ \infty) \\
&= \overline{0}_2
\end{aligned}$$

Mohri's example (here with $k = 3$)

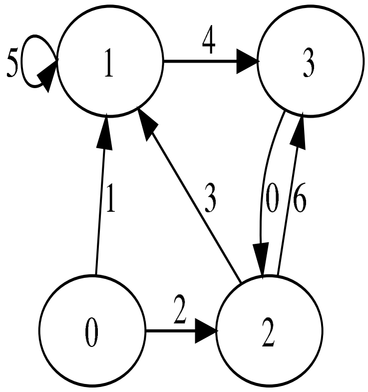


$$\mathbf{A} \;=\; \begin{array}{cc} & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} [\infty,\infty,\infty] & [1,\infty,\infty] & [2,\infty,\infty] & [\infty,\infty,\infty] \\ [\infty,\infty,\infty] & [5,\infty,\infty] & [\infty,\infty,\infty] & [4,\infty,\infty] \\ [\infty,\infty,\infty] & [3,\infty,\infty] & [\infty,\infty,\infty] & [6,\infty,\infty] \\ [\infty,\infty,\infty] & [\infty,\infty,\infty] & [0,\infty,\infty] & [\infty,\infty,\infty] \end{bmatrix} \end{array}$$

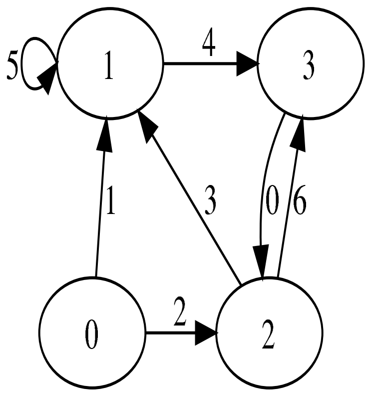
Red indicates change from previous iteration



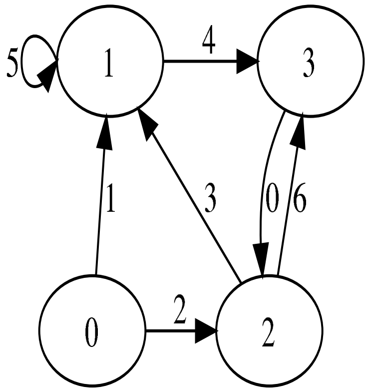
$$\mathbf{A}^{\langle 1 \rangle} \;=\; \begin{array}{cc} & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} [0,\infty,\infty] & [1,\infty,\infty] & [2,\infty,\infty] & [\infty,\infty,\infty] \\ [\infty,\infty,\infty] & [0,\textcolor{red}{5},\infty] & [\infty,\infty,\infty] & [4,\infty,\infty] \\ [\infty,\infty,\infty] & [3,\infty,\infty] & [0,\infty,\infty] & [6,\infty,\infty] \\ [\infty,\infty,\infty] & [\infty,\infty,\infty] & [0,\infty,\infty] & [0,\infty,\infty] \end{bmatrix} \end{array}$$



$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} [0, \infty, \infty] & [1, 5, 6] & [2, \infty, \infty] & [5, 8, \infty] \\ [\infty, \infty, \infty] & [0, 5, 10] & [4, \infty, \infty] & [4, 9, \infty] \\ [\infty, \infty, \infty] & [3, 8, \infty] & [0, 6, \infty] & [6, 7, \infty] \\ [\infty, \infty, \infty] & [3, \infty, \infty] & [0, \infty, \infty] & [0, 6, \infty] \end{array} \right] \end{array}$$

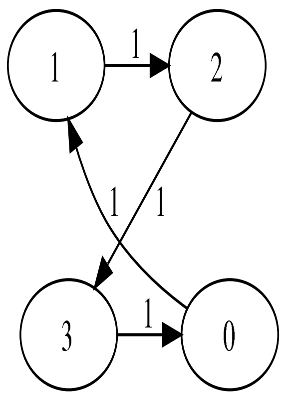


$$\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} [0, \infty, \infty] & [1, 5, 6] & [2, 5, 8] & [5, 8, 9] \\ [\infty, \infty, \infty] & [0, 5, 7] & [4, 9, \infty] & [4, 9, 10] \\ [\infty, \infty, \infty] & [3, 8, 9] & [0, 6, 7] & [6, 7, 12] \\ [\infty, \infty, \infty] & [3, 8, \infty] & [0, 6, \infty] & [0, 6, 7] \end{array} \right] \end{array}$$



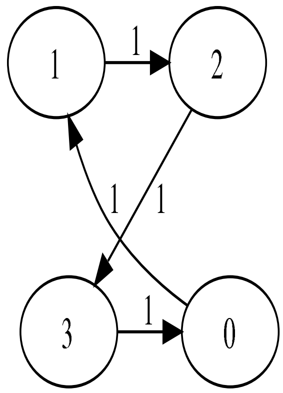
$$
\mathbf{A}^{\langle 4 \rangle} \quad = \quad
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\left[\begin{array}{cccc}
[0, \infty, \infty] & [1, 5, 6] & [2, 5, 8] & [5, 8, 9] \\
[\infty, \infty, \infty] & [0, 5, 7] & [4, 9, 10] & [4, 9, 10] \\
[\infty, \infty, \infty] & [3, 8, 9] & [0, 6, 7] & [6, 7, 12] \\
[\infty, \infty, \infty] & [3, 8, 9] & [0, 6, 7] & [0, 6, 7]
\end{array} \right]
\end{array}
$$

Another example : a simple cycle.



$$
\mathbf{A} \quad = \quad
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\left[\begin{array}{cccc}
[\infty, \infty, \infty] & [1, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty] \\
[\infty, \infty, \infty] & [\infty, \infty, \infty] & [1, \infty, \infty] & [\infty, \infty, \infty] \\
[\infty, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty] & [1, \infty, \infty] \\
[1, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty]
\end{array} \right]
\end{array}
$$

Solution $\mathbf{A}^*$ reached at 11-th iteration



$$\mathbf{A}^{\langle 11 \rangle} \quad = \quad \begin{array}{c c} & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} [0, 4, 8] & [1, 5, 9] & [2, 6, 10] & [3, 7, 11] \\ [3, 7, 11] & [0, 4, 8] & [1, 5, 9] & [2, 6, 10] \\ [2, 6, 10] & [3, 7, 11] & [0, 4, 8] & [1, 5, 9] \\ [1, 5, 9] & [2, 6, 10] & [3, 7, 11] & [0, 4, 8] \end{array} \right] \end{array}$$