L11: Algebraic Path Problems with applications to Internet Routing Lecture 11

Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK

Michaelmas Term, 2017

Path Weight with functions on arcs?

For graph $G = (V,\ E)$, and arc path $p = (u_0,\ u_1)(u_1,\ u_2)\cdots(u_{k-1},\ u_k)$.

Functions on arcs: two natural ways to do this...

Weight function $w : E \to (S \to S)$. Let $f_j = w(u_{j-1}, u_j)$.

$$\begin{array}{rclcl} w_a^L(p) & = & f_1(f_2(\cdots f_k(a)\cdots)) & = & (f_1 \circ f_2 \circ \cdots \circ f_k)(a) \\ w_a^R(p) & = & f_k(f_{k-1}(\cdots f_1(a)\cdots)) & = & (f_k \circ f_{k-1} \circ \cdots \circ f_1)(a) \end{array}$$

How can we “make this work” for path problems?

Algebra of Monoid Endomorphisms (AME) (See Gondran and Minoux 2008)

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \rightarrow S, \overline{0})$ is an algebra of monoid endomorphisms (AME) if

- $\forall f \in F, \; f(\overline{0}) = \overline{0}$
- $\forall f \in F, \; \forall b, c \in S, \; f(b \oplus c) = f(b) \oplus f(c)$

I will declare these as optional

- $\forall f, g \in F, \; f \circ g \in F$ (closed)
- $\exists i \in F, \; \forall s \in S, \; i(s) = s$
- $\exists \omega \in F, \; \forall n \in N, \; \omega(n) = \overline{0}$

Note: as with semirings, we may have to drop some of these axioms in order to model Internet routing ...

So why do we want AMEs?

Each (closed with ω and i) AME can be viewed as a semiring of functions. Suppose $(S, \oplus, F, \overline{0})$ is an algebra of monoid endomorphisms. We can turn it into a semiring

$$\mathbb{F} = (F, \hat{\oplus}, \circ, \omega, i)$$

where $(f \; \hat{\oplus} \; g)(a) = f(a) \oplus g(a)$ and $(f \circ g)(a) = f(g(a))$.

But functions are hard to work with....

- All algorithms need to check equality over elements of a semiring
- $f = g$ means $\forall a \in S, \; f(a) = g(a)$
- S can be very large, or infinite ....

How do we represent a set of functions $F \subseteq S \to S$?

Assume we a set L and a function

$$\rhd \in L \to (S \to S).$$

We normally write $l \rhd s$ rather than $\rhd(l)(s)$. We think of $l \in L$ as the index for a function $f_l(s) = l \rhd s$. In this way $(L, \rhd)$ can be used to represent the set of functions

$$F = \{f_l = \lambda s.(l \rhd s) \mid l \in L\}.$$

Indexed Algebra of Monoid Endomorphisms (IAME)

Let $(S, \oplus, \overline{0})$ be a commutative and idempotent monoid.

A (left) IAME $(S, L, \oplus, \rhd, \overline{0})$

- $\rhd \in L \to (S \to S)$
- $\forall l \in L, \ l \rhd \overline{0} = \overline{0}$
- $\exists l \in L, \ \forall s \in S, \ l \rhd s = s$
- $\exists l \in L, \ \forall s \in S, \ l \rhd s = \overline{0}$
- $\forall l \in L, \ \forall n, m \in S, \ l \rhd (n \oplus m) = (l \rhd n) \oplus (l \rhd m)$

When we need closure? Not very often! If needed, it would be

$$\forall l_1, l_2 \in L, \exists l_3 \in L, \forall s \in S, l_3 \rhd s = l_1 \rhd (l_2 \rhd s)$$

IAME of Matrices

Given a left IAME $(S,\ L,\ \oplus,\ \rhd,\ \overline{0})$ define the left IAME of matrices

$$(\mathbb{M}_n(S),\ \mathbb{M}_n(L),\ \oplus,\ \rhd,\ \mathbf{J}).$$

For all i, j we have $\mathbf{J}(i, j) = \overline{0}$. For $\mathbf{A} \in \mathbb{M}_n(L)$ and $\mathbf{B}, \mathbf{C} \in \mathbb{M}_n(S)$ define

$$\begin{aligned}
(\mathbf{B} \oplus \mathbf{C})(i,\ j) &= \mathbf{B}(i,\ j) \oplus \mathbf{C}(i,\ j) \\
(\mathbf{A} \rhd \mathbf{B})(i,\ j) &= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i,\ q) \rhd \mathbf{B}(q,\ j)
\end{aligned}$$

Solving (some) equations. Left version here ...

We will be interested in solving for $\mathbf{L}$ equations of the form

$$\mathbf{L} = (\mathbf{A} \rhd \mathbf{L}) \oplus \mathbf{B}$$

Let

$$\begin{aligned}
\mathbf{A} \rhd^{0} \mathbf{B} &= \mathbf{B} \\
\mathbf{A} \rhd^{k+1} \mathbf{B} &= \mathbf{A} \rhd (\mathbf{A} \rhd^{k} \mathbf{B})
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A} \rhd^{(k)} \mathbf{B} &= \mathbf{A} \rhd^{0} \mathbf{B} \oplus \mathbf{A} \rhd^{1} \mathbf{B} \oplus \mathbf{A} \rhd^{2} \mathbf{B} \oplus \cdots \oplus \mathbf{A} \rhd^{k} \mathbf{B} \\
\mathbf{A} \rhd^{*} \mathbf{B} &= \mathbf{A} \rhd^{0} \mathbf{B} \oplus \mathbf{A} \rhd^{1} \mathbf{B} \oplus \mathbf{A} \rhd^{2} \mathbf{B} \oplus \cdots \oplus \mathbf{A} \rhd^{k} \mathbf{B} \oplus \cdots
\end{aligned}$$

Key result (again)

q stability

If there exists a q such that for all **B**, $\mathbf{A} \rhd^{(q)} \mathbf{B} = \mathbf{A} \rhd^{(q+1)} \mathbf{B}$, then **A** is q-stable. Therefore, $\mathbf{A} \rhd^{*} \mathbf{B} = \mathbf{A} \rhd^{(q)} \mathbf{B}$.

Theorm

If **A** is q-stable, then $\mathbf{L} = \mathbf{A} \rhd^{*} (\mathbf{B})$ solves the equation

$$\mathbf{L} = (\mathbf{A} \rhd \mathbf{L}) \oplus \mathbf{B}.$$

Something familiar : Lexicographic product

$$(S,\ L_S,\ \oplus_S,\ \rhd_S)\vec{\times}(T,\ L_T,\ \oplus_T,\ \rhd_T) \equiv (S \times T,\ \ L_S \times L_T,\ \oplus_S \vec{\times} \oplus_T,\ \rhd_S \times \rhd_T)$$

Theorem

$$\begin{array}{c} \mathbb{D}((S,\ L_S,\ \oplus_S,\ \rhd_S)\ \vec{\times}\ (T,\ L_T,\ \oplus_T,\ \rhd_T)) \\ \iff \\ \mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T) \\ \wedge\ (\mathbb{C}(S,\ L_S,\ \rhd_S) \vee \mathbb{K}(T,\ L_T,\ \rhd_T)) \end{array}$$

Where

$$\begin{array}{rcl} \mathbb{D}(S,\ L,\ \oplus,\ \rhd) & \equiv & \forall a, b \in S,\ l \in L,\ l \rhd (a \oplus b) = (l \rhd a) \oplus (l \rhd b) \\ \mathbb{C}(S,\ L,\ \rhd) & \equiv & \forall a, b \in S,\ l \in L,\ l \rhd a = l \rhd b \implies a = b \\ \mathbb{K}(S,\ L,\ \rhd) & \equiv & \forall a, b \in S,\ l \in L,\ l \rhd a = l \rhd b \end{array}$$

Something new: Functional Union

$$(S,\ L_1,\ \oplus,\ \rhd_1) +_{\mathrm{m}} (S,\ L_2,\ \oplus,\ \rhd_2) = (S,\ L_1 \uplus L_2,\ \oplus,\ \rhd_1 \uplus \rhd_2)$$

Where

$$\begin{array}{rcl} \mathrm{inl}(l)\ (\rhd_1 \uplus \rhd_2)\ s & = & l\ \rhd_1\ s \\ \mathrm{inr}(l)\ (\rhd_1 \uplus \rhd_2)\ s & = & l\ \rhd_2\ s \end{array}$$

Fact

$$\mathbb{D}((S,\ L_1,\ \oplus,\ \rhd_1) +_{\mathrm{m}} (S,\ L_2,\ \oplus,\ \rhd_2))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S,\ L_1,\ \oplus,\ \rhd_1) \wedge \mathbb{D}(S,\ L_2,\ \oplus,\ \rhd_2)$$

Left and Right

$\mathrm{right}(S,\ \oplus) \equiv (S,\ \{R\},\ \oplus, \textbf{right})$

$$R\ \textbf{right}\ s = s$$

$\mathrm{left}(S,\ \oplus) \equiv (S,\ S,\ \oplus,\ \textbf{left})$

$$s_1\ \textbf{left}\ s_2 = s_1$$

The following are always hold.

$$\mathbb{D}(\mathrm{right}(S,\ \oplus))$$
$$\mathbb{IP}(S,\ \oplus) \Rightarrow \mathbb{D}(\mathrm{left}(S,\ \oplus))$$
$$\mathbb{C}(\mathrm{right}(S,\ \oplus))$$
$$\mathbb{K}(\mathrm{left}(S,\ \oplus))$$

Scoped Product (Think iBGP/eBGP)

$$(S,\ L_S,\ \oplus_S,\ \rhd_S) \circleddash (T,\ L_T,\ \oplus_T,\ \rhd_T)$$
$$\equiv$$
$$((S,\ L_S,\ \oplus_S,\ \rhd_S) \vec{\times} \operatorname{left}(T,\ \oplus_T)) +_{\mathrm{m}} (\operatorname{right}(S,\ \oplus_S) \vec{\times} (T,\ L_T,\ \oplus_T,\ \rhd_T))$$

Between regions ($\lambda \in L_S$, $s \in S$, $t_1, t_2 \in T$)

$$\operatorname{inl}(\lambda,\ t_2) \rhd (s,\ t_1) = (\lambda \rhd_S s,\ t_2)$$

Within regions ($\lambda \in L_T$, $s \in S$, $t \in T$)

$$\operatorname{inr}(R,\ \lambda) \rhd (s,\ t) = (s,\ \lambda \rhd_T t)$$

Theorem. If $\mathbb{IP}(T,\ \oplus_T)$, then

$$(\mathbb{D}((S,\ L_S,\ \oplus_S,\ \rhd_S) \circleddash (T,\ L_T,\ \oplus_T,\ \rhd_T))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T))$$

$$\begin{aligned}
& \mathbb{D}(((S,\ L_S,\ \oplus_S,\ \rhd_S) \vec{\times} \operatorname{left}(T,\ \oplus_T)) \\
& \qquad +_{\mathrm{m}} (\operatorname{right}(S,\ \oplus_S) \vec{\times} (T,\ L_T,\ \oplus_T,\ \rhd_T))) \\
\Longleftrightarrow\ & \mathbb{D}((S,\ L_S,\ \oplus_S,\ \rhd_S) \vec{\times} \operatorname{left}(T,\ \oplus_T)) \\
& \qquad \wedge\ \mathbb{D}((\operatorname{right}(S,\ \oplus_S)) \vec{\times} (T,\ L_T,\ \oplus_T,\ \rhd_T)) \\
\Longleftrightarrow\ & \mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(\operatorname{left}(T,\ \oplus_T)) \\
& \qquad \wedge\ (\mathbb{C}(S,\ L_S,\ \rhd_S) \vee \mathbb{K}(\operatorname{left}(T,\ \oplus_T))) \\
& \qquad \wedge\ \mathbb{D}(\operatorname{right}(S,\ \oplus_S)) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T) \\
& \qquad \wedge\ (\mathbb{C}(\operatorname{right}(S,\ \oplus_S)) \vee \mathbb{K}(T,\ L_T,\ \rhd_T)) \\
\Longleftrightarrow\ & \mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T)
\end{aligned}$$