# L11: Algebraic Path Problems with applications to Internet Routing Lectures 01 – 03


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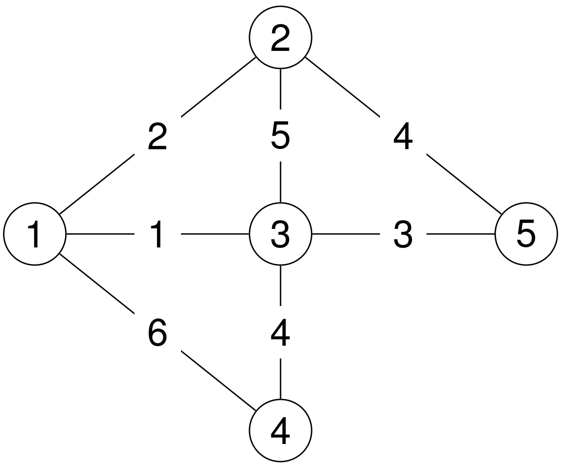
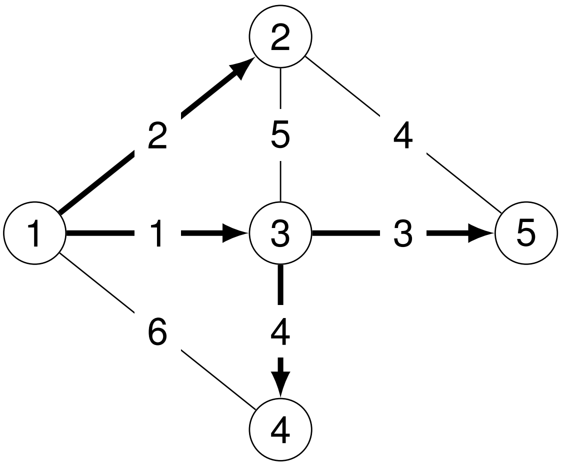
## Shortest paths example, $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +, \infty,\ 0)$




The adjacency matrix

$$
\mathbf{A} \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
\infty & 2 & 1 & 6 & \infty \\
2 & \infty & 5 & \infty & 4 \\
1 & 5 & \infty & 4 & 3 \\
6 & \infty & 4 & \infty & \infty \\
\infty & 4 & 3 & \infty & \infty
\end{array}\right]
\end{array}
$$

## Shortest paths solution




$$\mathbf{A}^* \quad = \quad \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

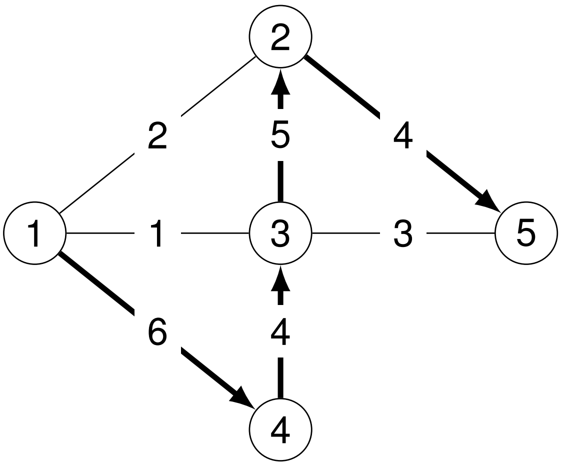
solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from $i$ to $j$.



## Widest paths example, $\mathrm{bw} = (\mathbb{N}^\infty,\ \max,\ \min,\ 0,\ \infty)$



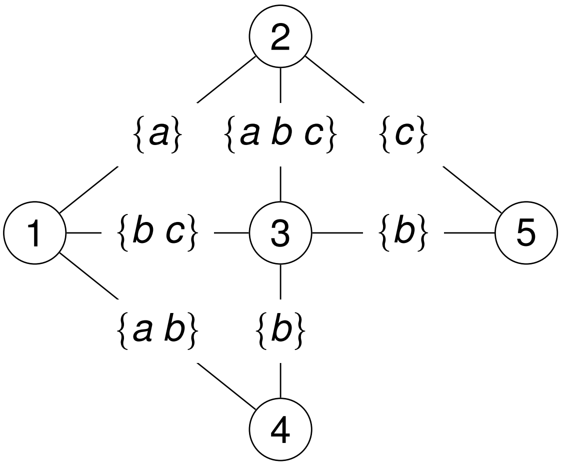

$$\mathbf{A}^* \quad = \quad \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{bmatrix} \end{array}$$

solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \max_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the minimal edge weight in $p$.

## Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap, \{\},\ \{a,\ b,\ c\})$




We want $\mathbf{A}^*$ to solve this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

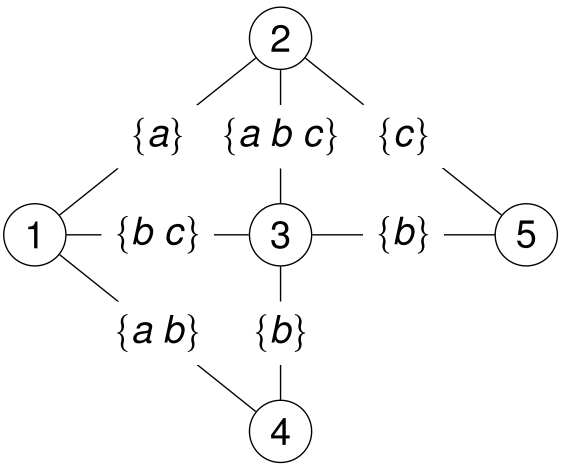
where $w(p)$ is now the intersection of all edge weights in $p$.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{A}^*(i,\ j)$ to mean that there is at least one path from $i$ to $j$ with $x$ in every arc weight along the path.

$$\mathbf{A}^*(4,\ 1) = \{a,\ b\} \qquad \mathbf{A}^*(4,\ 5) = \{b\}$$



## Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$




We want matrix $\mathbf{R}$ to solve this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \bigcap_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the union of all edge weights in $p$.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that every path from $i$ to $j$ has at least one arc with weight containing $x$.

$$\mathbf{A}^*(4,\ 1) = \{b\} \qquad \mathbf{A}^*(4,\ 5) = \{b\} \qquad \mathbf{A}^*(5,\ 1) = \{\}$$

# Semirings (generalise $(\mathbb{R}, +, \times, 0, 1)$)

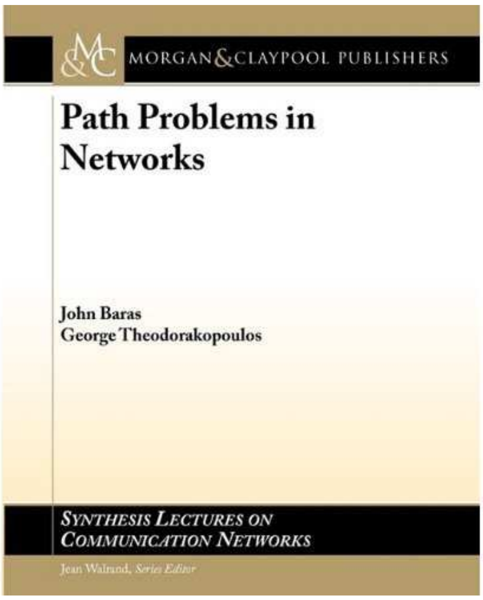
| name | $S$ | $\oplus$, | $\otimes$ | $\overline{0}$ | $\overline{1}$ | possible routing use |
|---|---|---|---|---|---|---|
| sp | $\mathbb{N}^{\infty}$ | min | $+$ | $\infty$ | 0 | minimum-weight routing |
| bw | $\mathbb{N}^{\infty}$ | max | min | 0 | $\infty$ | greatest-capacity routing |
| rel | $[0,\ 1]$ | max | $\times$ | 0 | 1 | most-reliable routing |
| use | $\{0,\ 1\}$ | max | min | 0 | 1 | usable-path routing |
| | $2^{W}$ | $\cup$ | $\cap$ | $\{\}$ | $W$ | shared link attributes? |
| | $2^{W}$ | $\cap$ | $\cup$ | $W$ | $\{\}$ | shared path attributes? |

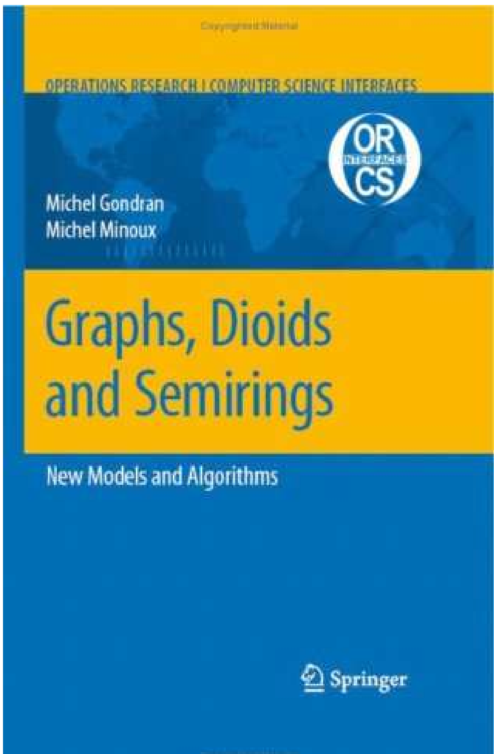
## A wee bit of notation!

| Symbol | Interpretation |
|---|---|
| $\mathbb{N}$ | Natural numbers (starting with zero) |
| $\mathbb{N}^{\infty}$ | Natural numbers, plus infinity |
| $\overline{0}$ | Identity for $\oplus$ |
| $\overline{1}$ | Identity for $\otimes$ |



# Recommended (on reserve in CL library)



## Semiring axioms ...

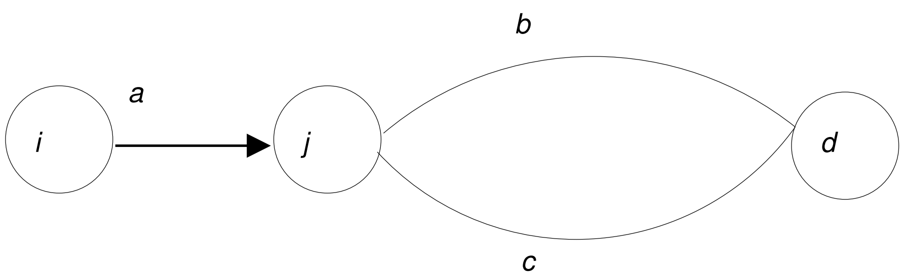
We will look at all of the axioms of semirings, but the most important are

**distributivity**

$$\begin{array}{lcrcl} \mathbb{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\ \mathbb{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c) \end{array}$$



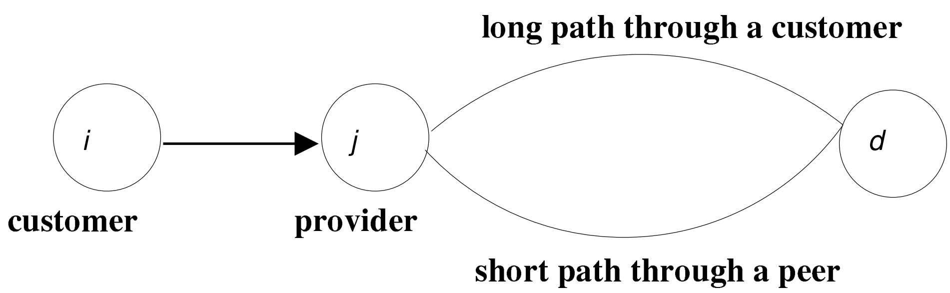
## Distributivity, illustrated




$$a \otimes (b \oplus c) \quad = \quad (a \otimes b) \oplus (a \otimes c)$$

$j$ makes the choice $=$ $i$ makes the choice

## Should distributivity hold in Internet Routing?



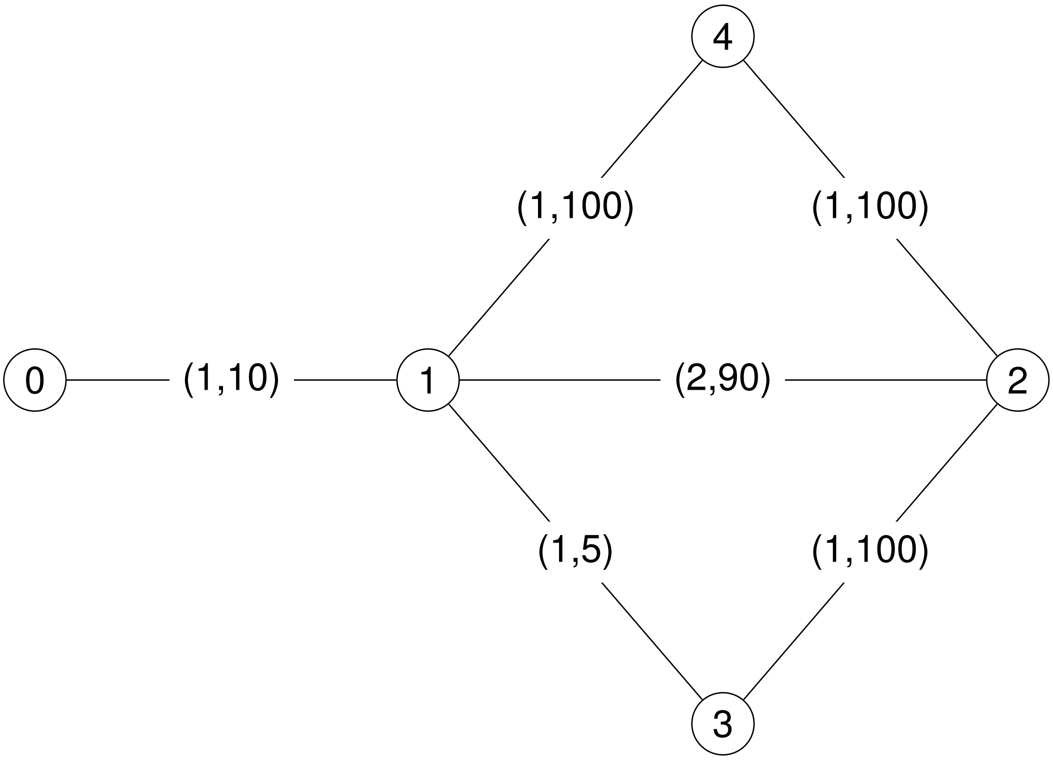

- $j$ prefers long path though one of its customers (not the shorter path through a competitor)
- given two routes from a provider, $i$ prefers the one with a shorter path
- More on inter-domain routing in the Internet later in the term ...



## Widest shortest-paths

- Metric of the form $(d, \ b)$, where $d$ is distance $(\min, \ +)$ and $b$ is capacity $(\max, \ \min)$.
- Metrics are compared lexicographically, with distance considered first.
- Such things are found in the vast literature on Quality-of-Service (QoS) metrics for Internet routing.

# Widest shortest-paths






# Weights are globally optimal (we have a semiring)

## Widest shortest-path weights computed by Dijkstra and Bellman-Ford

$$\mathbf{R} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} (0,\infty) & (1,10) & (3,10) & (2,5) & (2,10) \\ (1,10) & (0,\infty) & (2,100) & (1,5) & (1,100) \\ (3,10) & (2,100) & (0,\infty) & (1,100) & (1,100) \\ (2,5) & (1,5) & (1,100) & (0,\infty) & (2,100) \\ (2,10) & (1,100) & (1,100) & (2,100) & (0,\infty) \end{bmatrix} \end{array}$$

## But what about the paths themselves?

**Four optimal paths of weight $(3, 10)$.**

$$\begin{array}{lcl} \mathbf{P}_{\text{optimal}}(0,2) & = & \{(0,1,2),\ (0,1,4,2)\} \\ \mathbf{P}_{\text{optimal}}(2,0) & = & \{(2,1,0),\ (2,4,1,0)\} \end{array}$$

There are standard ways to extend Bellman-Ford and Dijkstra to compute paths (or the associated next hops).

Do these extended algorithms find all optimal paths?



## Surprise!

**Four optimal paths of weight $(3, 10)$**

$$\begin{array}{lcl} \mathbf{P}_{\text{optimal}}(0,2) & = & \{(0,1,2),\ (0,1,4,2)\} \\ \mathbf{P}_{\text{optimal}}(2,0) & = & \{(2,1,0),\ (2,4,1,0)\} \end{array}$$

**Paths computed by (extended) Dijkstra**

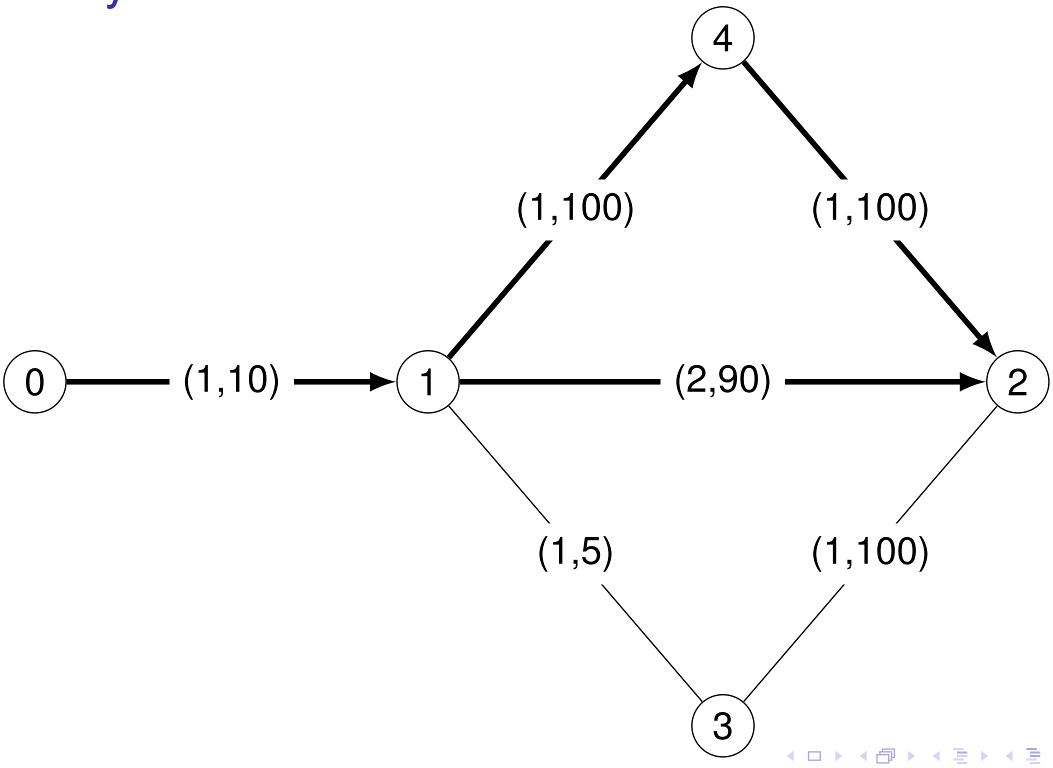
$$\begin{array}{lcl} \mathbf{P}_{\text{Dijkstra}}(0,2) & = & \{(0,1,2),\ (0,1,4,2)\} \\ \mathbf{P}_{\text{Dijkstra}}(2,0) & = & \{(2,4,1,0)\} \end{array}$$

Notice that 0's paths cannot both be implemented with next-hop forwarding since $\mathbf{P}_{\text{Dijkstra}}(1,2) = \{(1,4,2)\}$.

**Paths computed by distributed Bellman-Ford**

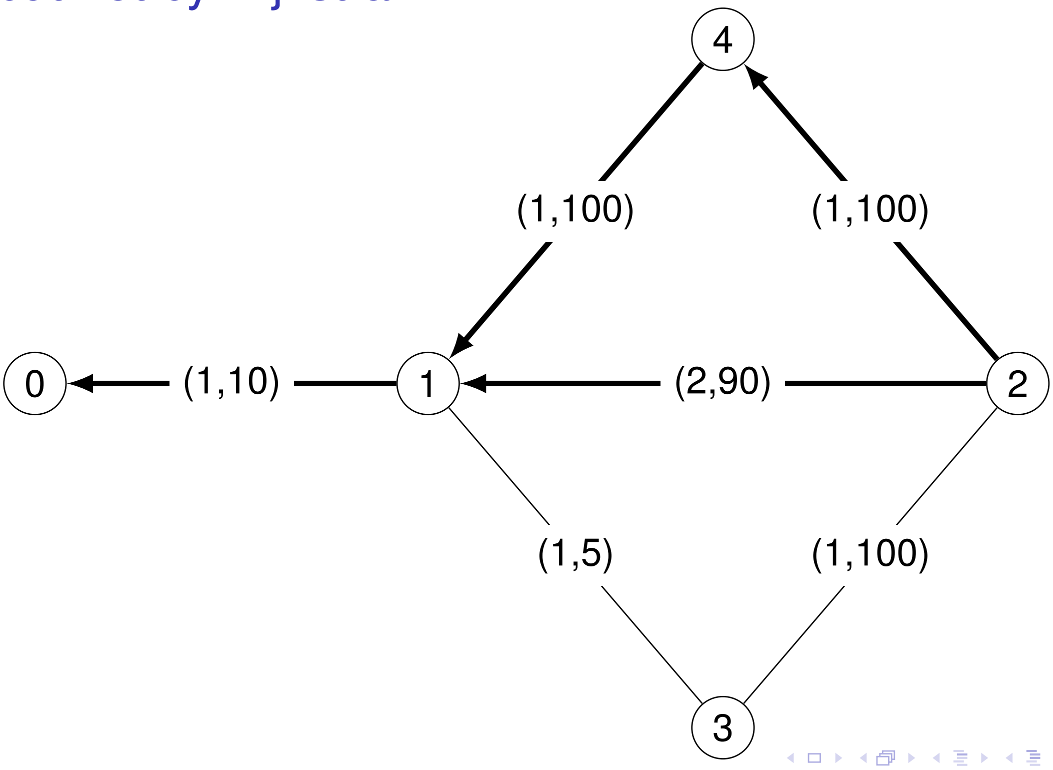
$$\begin{array}{lcl} \mathbf{P}_{\text{Bellman}}(0,2) & = & \{(0,1,4,2)\} \\ \mathbf{P}_{\text{Bellman}}(2,0) & = & \{(2,1,0),\ (2,4,1,0)\} \end{array}$$

## Optimal paths from 0 to 2. Computed by Dijkstra but not by Bellman-Ford






## Optimal paths from 2 to 1. Computed by Bellman-Ford but not by Dijkstra



## How can we understand this (algebaically)?

### The Algorithm to Algebra (A2A) method

$$\left(\begin{array}{c}\text{original metric}\\ +\\ \text{complex algorithm}\end{array}\right) \rightarrow \left(\begin{array}{c}\text{modified metric}\\ +\\ \text{matrix equations (generic algorithm)}\end{array}\right)$$

### Preview

- We can add paths explicitly to the widest shortest-path semiring to obtain a new algebra.
- We will see that distributivity does not hold for this algebra.
- Why? We will see that it is because min is not cancellative! ($a \min b = a \min c$ does not imply that $b = c$)



## Towards a non-classical theory of algebraic path finding

We need theory that can accept algebras that violate distributivity.

### Global optimality

$$\mathbf{A}^*(i,\ j) = \bigoplus_{p \in P(i,\ j)} w(p),$$

### Left local optimality (distributed Bellman-Ford)

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}.$$

### Right local optimality (Dijkstra's Algorithm)

$$\mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}.$$

Embrace the fact that all three notions can be distinct.

## Assessment

Five homeworks, with only best four counted, each 25%.

| | due |
|---|---|
| 1 | October 16 |
| 2 | October 27 |
| 3 | November 6 |
| 4 | November 17 |
| 5 | December 1 |



## Lectures 2, 3

- Semigroups
- A few important semigroup properties
- Semigroup and partial orders

## Semigroups

### Semigroup

A semigroup $(S, \bullet)$ is a non-empty set $S$ with a binary operation such that

$$\mathbb{AS} \quad \text{associative} \quad \equiv \quad \forall a, b, c \in S, \ a \bullet (b \bullet c) = (a \bullet b) \bullet c$$

### Important Assumption — We will ignore trival semigroups

We will impicitly assume that $2 \leqslant\mid S \mid$.

### Note

Many useful binary operations are not semigroup operations. For example, $(\mathbb{R}, \bullet)$, where $a \bullet b \equiv (a + b)/2$.



## Some Important Semigroup Properties

$$\begin{array}{llcl}
\mathbb{ID} & \text{identity} & \equiv & \exists \alpha \in S, \ \forall a \in S, \ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{AN} & \text{annihilator} & \equiv & \exists \omega \in S, \ \forall a \in S, \ \omega = \omega \bullet a = a \bullet \omega \\
\mathbb{CM} & \text{commutative} & \equiv & \forall a, b \in S, \ a \bullet b = b \bullet a \\
\mathbb{SL} & \text{selective} & \equiv & \forall a, b \in S, \ a \bullet b \in \{a, \ b\} \\
\mathbb{IP} & \text{idempotent} & \equiv & \forall a \in S, \ a \bullet a = a
\end{array}$$

A semigroup with an identity is called a monoid.

### Note that

$$\mathbb{SL}(S, \ \bullet) \implies \mathbb{IP}(S, \ \bullet)$$

## A few concrete semigroups

| $S$ | $\bullet$ | description | $\alpha$ | $\omega$ | $\mathbb{CM}$ | $\mathbb{SL}$ | $\mathbb{IP}$ |
|---|---|---|---|---|---|---|---|
| $S$ | left | $x$ left $y = x$ | | | | $\star$ | $\star$ |
| $S$ | right | $x$ right $y = y$ | | | | $\star$ | $\star$ |
| $S^*$ | $\cdot$ | concatenation | $\epsilon$ | | | | |
| $S^+$ | $\cdot$ | concatenation | | | | | |
| $\{t, f\}$ | $\wedge$ | conjunction | t | f | $\star$ | $\star$ | $\star$ |
| $\{t, f\}$ | $\vee$ | disjunction | f | t | $\star$ | $\star$ | $\star$ |
| $\mathbb{N}$ | min | minimum | | 0 | $\star$ | $\star$ | $\star$ |
| $\mathbb{N}$ | max | maximum | 0 | | $\star$ | $\star$ | $\star$ |
| $2^W$ | $\cup$ | union | $\{\}$ | $W$ | $\star$ | | $\star$ |
| $2^W$ | $\cap$ | intersection | $W$ | $\{\}$ | $\star$ | | $\star$ |
| $\mathrm{fin}(2^U)$ | $\cup$ | union | $\{\}$ | | $\star$ | | $\star$ |
| $\mathrm{fin}(2^U)$ | $\cap$ | intersection | | $\{\}$ | $\star$ | | $\star$ |
| $\mathbb{N}$ | $+$ | addition | 0 | | $\star$ | | |
| $\mathbb{N}$ | $\times$ | multiplication | 1 | 0 | $\star$ | | |

$W$ a finite set, $U$ an infinite set. For set $Y$, $\mathrm{fin}(Y) \equiv \{X \in Y \mid X \text{ is finite}\}$



## A few abstract semigroups

| $S$ | $\bullet$ | description | $\alpha$ | $\omega$ | $\mathbb{CM}$ | $\mathbb{SL}$ | $\mathbb{IP}$ |
|---|---|---|---|---|---|---|---|
| $2^U$ | $\cup$ | union | $\{\}$ | $U$ | $\star$ | | $\star$ |
| $2^U$ | $\cap$ | intersection | $U$ | $\{\}$ | $\star$ | | $\star$ |
| $2^{U \times U}$ | $\bowtie$ | relational join | $\mathcal{I}_U$ | $\{\}$ | | | |
| $X \rightarrow X$ | $\circ$ | composition | $\lambda x.x$ | | | | |

$U$ an infinite set

$X \bowtie Y \equiv \{(x, z) \in U \times U \mid \exists y \in U,\ (x,\ y) \in X \wedge (y,\ z) \in Y\}$

$\mathcal{I}_U \equiv \{(u,\ u) \mid u \in U\}$

### subsemigroup

Suppose $(S,\ \bullet)$ is a semigroup and $T \subseteq S$. If $T$ is closed w.r.t $\bullet$ (that is, $\forall x, y \in T, x \bullet y \in T$), then $(T,\ \bullet)$ is a subsemigroup of $S$.

## Order Relations

We are interested in order relations $\leqslant \subseteq S \times S$

### Definition (Important Order Properties)

$$\begin{array}{llcl} \mathbb{RX} & \text{reflexive} & \equiv & a \leqslant a \\ \mathbb{TR} & \text{transitive} & \equiv & a \leqslant b \wedge b \leqslant c \rightarrow a \leqslant c \\ \mathbb{AY} & \text{antisymmetric} & \equiv & a \leqslant b \wedge b \leqslant a \rightarrow a = b \\ \mathbb{TO} & \text{total} & \equiv & a \leqslant b \vee b \leqslant a \end{array}$$

| | pre-order | partial order | preference order | total order |
|---|---|---|---|---|
| $\mathbb{RX}$ | $\star$ | $\star$ | $\star$ | $\star$ |
| $\mathbb{TR}$ | $\star$ | $\star$ | $\star$ | $\star$ |
| $\mathbb{AY}$ | | $\star$ | | $\star$ |
| $\mathbb{TO}$ | | | $\star$ | $\star$ |



## Canonical Pre-order of a Commutative Semigroup

### Definition (Canonical pre-orders)

$$\begin{array}{lcl} a \trianglelefteq_{\bullet}^{R} b & \equiv & \exists c \in S : b = a \bullet c \\ a \trianglelefteq_{\bullet}^{L} b & \equiv & \exists c \in S : a = b \bullet c \end{array}$$

### Lemma (Sanity check)

*Associativity of $\bullet$ implies that these relations are transitive.*

### Proof.

Note that $a \trianglelefteq_{\bullet}^{R} b$ means $\exists c_1 \in S : b = a \bullet c_1$, and $b \trianglelefteq_{\bullet}^{R} c$ means $\exists c_2 \in S : c = b \bullet c_2$. Letting $c_3 = c_1 \bullet c_2$ we have $c = b \bullet c_2 = (a \bullet c_1) \bullet c_2 = a \bullet (c_1 \bullet c_2) = a \bullet c_3$. That is, $\exists c_3 \in S : c = a \bullet c_3$, so $a \trianglelefteq_{\bullet}^{R} c$. The proof for $\trianglelefteq_{\bullet}^{L}$ is similar. $\square$

# Canonically Ordered Semigroup

## Definition (Canonically Ordered Semigroup)

A commutative semigroup $(S, \bullet)$ is canonically ordered when $a \trianglelefteq^R_\bullet c$ and $a \trianglelefteq^L_\bullet c$ are partial orders.

## Definition (Groups)

A monoid is a group if for every $a \in S$ there exists a $a^{-1} \in S$ such that $a \bullet a^{-1} = a^{-1} \bullet a = \alpha$.



# Canonically Ordered Semigroups vs. Groups

## Lemma (THE BIG DIVIDE)

*Only a trivial group is canonically ordered.*

## Proof.

If $a, b \in S$, then $a = \alpha_\bullet \bullet a = (b \bullet b^{-1}) \bullet a = b \bullet (b^{-1} \bullet a) = b \bullet c$, for $c = b^{-1} \bullet a$, so $a \trianglelefteq^L_\bullet b$. In a similar way, $b \trianglelefteq^R_\bullet a$. Therefore $a = b$. $\square$

## Natural Orders

### Definition (Natural orders)

Let $(S, \bullet)$ be a semigroup.

$$
\begin{array}{lcl}
a \leqslant^L_\bullet b & \equiv & a = a \bullet b \\
a \leqslant^R_\bullet b & \equiv & b = a \bullet b
\end{array}
$$

### Lemma

*If $\bullet$ is commutative and idempotent, then $a \trianglelefteq^D_\bullet b \iff a \leqslant^D_\bullet b$, for $D \in \{R, L\}$.*

### Proof.

$$
\begin{array}{rcl}
a \trianglelefteq^R_\bullet b & \iff & b = a \bullet c = (a \bullet a) \bullet c = a \bullet (a \bullet c) \\
 & = & a \bullet b \iff a \leqslant^R_\bullet b \\
a \trianglelefteq^L_\bullet b & \iff & a = b \bullet c = (b \bullet b) \bullet c = b \bullet (b \bullet c) \\
 & = & b \bullet a = a \bullet b \iff a \leqslant^L_\bullet b
\end{array}
$$

$\square$



## Special elements and natural orders

### Lemma (Natural Bounds)

- *If $\alpha$ exists, then for all a, $a \leqslant^L_\bullet \alpha$ and $\alpha \leqslant^R_\bullet a$*
- *If $\omega$ exists, then for all a, $\omega \leqslant^L_\bullet a$ and $a \leqslant^R_\bullet \omega$*
- *If $\alpha$ and $\omega$ exist, then S is bounded.*

$$
\begin{array}{ccccc}
\omega & \leqslant^L_\bullet & a & \leqslant^L_\bullet & \alpha \\
\alpha & \leqslant^R_\bullet & a & \leqslant^R_\bullet & \omega
\end{array}
$$

### Remark (Thanks to Iljitsch van Beijnum)

Note that this means for $(\min, +)$ we have

$$
\begin{array}{ccccc}
0 & \leqslant^L_{\min} & a & \leqslant^L_{\min} & \infty \\
\infty & \leqslant^R_{\min} & a & \leqslant^R_{\min} & 0
\end{array}
$$

and still say that this is bounded, even though one might argue with the terminology!

# Examples of special elements

| $S$ | $\bullet$ | $\alpha$ | $\omega$ | $\leqslant^{\mathrm{L}}_{\bullet}$ | $\leqslant^{\mathrm{R}}_{\bullet}$ |
|---|---|---|---|---|---|
| $\mathbb{N}^{\infty}$ | min | $\infty$ | 0 | $\leqslant$ | $\geqslant$ |
| $\mathbb{N}^{-\infty}$ | max | 0 | $-\infty$ | $\geqslant$ | $\leqslant$ |
| $\mathcal{P}(W)$ | $\cup$ | $\{\}$ | $W$ | $\subseteq$ | $\supseteq$ |
| $\mathcal{P}(W)$ | $\cap$ | $W$ | $\{\}$ | $\supseteq$ | $\subseteq$ |



# Property Management

**Lemma**

*Let $D \in \{R, L\}$.*

1. $\mathbb{IP}(S, \bullet) \iff \mathbb{RX}(S, \leqslant^{D}_{\bullet})$
2. $\mathbb{CM}(S, \bullet) \implies \mathbb{AY}(S, \leqslant^{D}_{\bullet})$
3. $\mathbb{AS}(S, \bullet) \implies \mathbb{TR}(S, \leqslant^{D}_{\bullet})$
4. $\mathbb{CM}(S, \bullet) \implies (\mathbb{SL}(S, \bullet) \iff \mathbb{TO}(S, \leqslant^{D}_{\bullet}))$

**Proof.**

1. $a \leqslant^{D}_{\bullet} a \iff a = a \bullet a,$
2. $a \leqslant^{L}_{\bullet} b \wedge b \leqslant^{L}_{\bullet} a \iff a = a \bullet b \wedge b = b \bullet a \implies a = b$
3. $a \leqslant^{L}_{\bullet} b \wedge b \leqslant^{L}_{\bullet} c \iff a = a \bullet b \wedge b = b \bullet c \implies a = a \bullet (b \bullet c) = (a \bullet b) \bullet c = a \bullet c \implies a \leqslant^{L}_{\bullet} c$
4. $a = a \bullet b \vee b = a \bullet b \iff a \leqslant^{L}_{\bullet} b \vee b \leqslant^{L}_{\bullet} a$

$\square$

## Bounds

Suppose $(S, \leqslant)$ is a partially ordered set.

**greatest lower bound**

For $a, b \in S$, the element $c \in S$ is the greatest lower bound of $a$ and $b$, written $c = a \operatorname{glb} b$, if it is a lower bound ($c \leqslant a$ and $c \leqslant b$), and for every $d \in S$ with $d \leqslant a$ and $d \leqslant b$, we have $d \leqslant c$.

**least upper bound**

For $a, b \in S$, the element $c \in S$ is the least upper bound of $a$ and $b$, written $c = a \operatorname{lub} b$, if it is an upper bound ($a \leqslant c$ and $b \leqslant c$), and for every $d \in S$ with $a \leqslant d$ and $b \leqslant d$, we have $c \leqslant d$.



## Semi-lattices

Suppose $(S, \leqslant)$ is a partially ordered set.

**meet-semilattice**

$S$ is a meet-semilattice if $a \operatorname{glb} b$ exists for each $a, b \in S$.

**join-semilattice**

$S$ is a join-semilattice if $a \operatorname{lub} b$ exists for each $a, b \in S$.

## Fun Facts

### Fact 1

Suppose $(S, \bullet)$ is a commutative and idempotent semigroup.

- $(S, \leqslant^L_\bullet)$ is a meet-semilattice with $a \,\mathrm{glb}\, b = a \bullet b$.
- $(S, \leqslant^R_\bullet)$ is a join-semilattice with $a \,\mathrm{lub}\, b = a \bullet b$.

### Fact 2

Suppose $(S, \leqslant)$ is a partially ordered set.

- If $(S, \leqslant)$ is a meet-semilattice, then $(S, \mathrm{glb})$ is a commutative and idempotent semigroup.
- If $(S, \leqslant)$ is a join-semilattice, then $(S, \mathrm{lub})$ is a commutative and idempotent semigroup.

That is, semi-lattices represent the same class of structures as commutative and idempotent semigroups.



## Homework 1 (due 16 October)

Prove Fun Fun Facts 1 and 2.