Appendix C: NURBS

- **NURBS** ("Non-Uniform Rational B-Splines") are a generalization of Bezier curves.
  - **NU**: Non-Uniform. The knots in the knot vector are not required to be uniformly spaced.
  - **R**: Rational. The spline may be defined by rational polynomials (homogeneous coordinates.)
  - **BS**: B-Spline. A generalized Bezier spline with controllable degree.
B-Splines

We’ll build our definition of a B-spline from:

- $d$, the *degree* of the curve
- $k = d+1$, called the *parameter* of the curve
- $\{P_1...P_n\}$, a list of $n$ *control points*
- $[t_1,...,t_{k+n}]$, a *knot vector* of $(k+n)$ parameter values ("knots")
- $d = k-1$ is the degree of the curve, so $k$ is the number of control points which influence a single interval.
  - Ex: a cubic ($d=3$) has four control points ($k=4$).
- There are $k+n$ knots $t_i$, and $t_i \leq t_{i+1}$ for all $t_i$.
- Each B-spline is $C^{(k-2)}$ continuous: *continuity* is degree minus one, so a $k=3$ curve has $d=2$ and is $C1$. 
B-Splines

- The equation for a B-spline curve is
  \[ P(t) = \sum_{i=1}^{n} N_{i,k}(t) P_i, \quad t_{\text{min}} \leq t < t_{\text{max}} \]

- \( N_{i,k}(t) \) is the \textit{basis function} of control point \( P_i \) for parameter \( k \). \( N_{i,k}(t) \) is defined recursively:

  \[
  N_{i,1}(t) = \begin{cases} 
  1, & t_i \leq t < t_{i+1} \\
  0, & \text{otherwise}
  \end{cases}
  \]

  \[
  N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)
  \]
B-Splines

\[ N_{1,1}(t) \quad N_{2,1}(t) \quad N_{3,1}(t) \quad N_{4,1}(t) \quad \ldots \]

\[ N_{1,2}(t) \quad N_{2,2}(t) \quad N_{3,2}(t) \quad \ldots \]

\[ N_{1,3}(t) \quad N_{2,3}(t) \quad \ldots \]

\[ N_{1,4}(t) \quad \ldots \]
B-Splines

$$N_{i,1}(t) = \begin{cases} 
1, & t_i \leq t < t_{i+1} \\
0, & \text{otherwise} 
\end{cases}$$

Knot vector = \{0,1,2,3,4,5\}, \(k = 1 \rightarrow d = 0\) (degree = zero)
B-Splines

\[
N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)
\]

Knot vector = \{0,1,2,3,4,5\}, \ k = 2 \rightarrow d = 1 \ (\text{degree} = \text{one})
B-Splines

\[
N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)
\]

Knot vector = \{0,1,2,3,4,5\}, \ k = 3 \rightarrow d = 2 \ (\text{degree} = \text{two})
Basis functions really sum to one (k=2)

The sum of the four basis functions is fully defined (sums to one) between $t_2$ ($t=1.0$) and $t_5$ ($t=4.0$).
Basis functions really sum to one (k=3)

The sum of the three functions is fully defined (sums to one) between $t_3$ ($t=2.0$) and $t_4$ ($t=3.0$).
B-Splines

At $k=2$ the function is piecewise linear, depends on $P_1, P_2, P_3, P_4$, and is fully defined on $[t_2, t_5)$.

Each parameter-$k$ basis function depends on $k+1$ knot values; $N_{i,k}$ depends on $t_i$ through $t_{i+k}$, inclusive. So six knots $\rightarrow$ five discontinuous functions $\rightarrow$ four piecewise linear interpolations $\rightarrow$ three quadratics, interpolating three control points. $n=3$ control points, $d=2$ degree, $k=3$ parameter, $n+k=6$ knots.

At $k=3$ the function is piecewise quadratic, depends on $P_1, P_2, P_3$, and is fully defined on $[t_3, t_4)$.

Knot vector $= \{0,1,2,3,4,5\}$
Non-Uniform B-Splines

- The knot vector \{0,1,2,3,4,5\} is \textit{uniform}:
  \[
t_{i+1} - t_i = t_{i+2} - t_{i+1} \forall t_i.
  \]
- Varying the size of an interval changes the parametric-space distribution of the weights assigned to the control functions.
- Repeating a knot value reduces the continuity of the curve in the affected span by one degree.
- Repeating a knot \(k\) times will lead to a control function being influenced only by that knot value; the spline will pass through the corresponding control point with C0 continuity.
Open vs Closed

- A knot vector which repeats its first and last knot values $k$ times is called open, otherwise closed.
- Repeating the knots $k$ times is the only way to force the curve to pass through the first or last control point.
- Without this, the functions $N_{1,k}$ and $N_{n,k}$ which weight $P_1$ and $P_n$ would still be ‘ramping up’ and not yet equal to one at the first and last $t_i$. 
Open vs Closed

- Two examples you may recognize:
  - $k=3$, $n=3$ control points, knots=$\{0,0,0,1,1,1\}$
  - $k=4$, $n=4$ control points, knots=$\{0,0,0,0,1,1,1,1\}$
Non-Uniform *Rational* B-Splines

- Repeating knot values is a clumsy way to control the curve’s proximity to the control point.
  - The solution: *homogeneous coordinates*.
  - Associate a ‘weight’ with each control point, $\omega_i$, so that the expression becomes a weighted average.
  - This allows us to slide the curve nearer or farther to individual control points without losing continuity or introducing new control points.
Non-Uniform Rational B-Splines in action

Weights

Control functions

Spline

Demo
NURBS - References