Theorem 134 *For every set* A,

$$\begin{tikzcd} \mathrm{EqRel}(A) \arrow[r, bend left, "\pi"] & \mathrm{Part}(A) \arrow[l, bend left, "\mathcal{E}"] \end{tikzcd}$$

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) \quad .$$

PROOF:

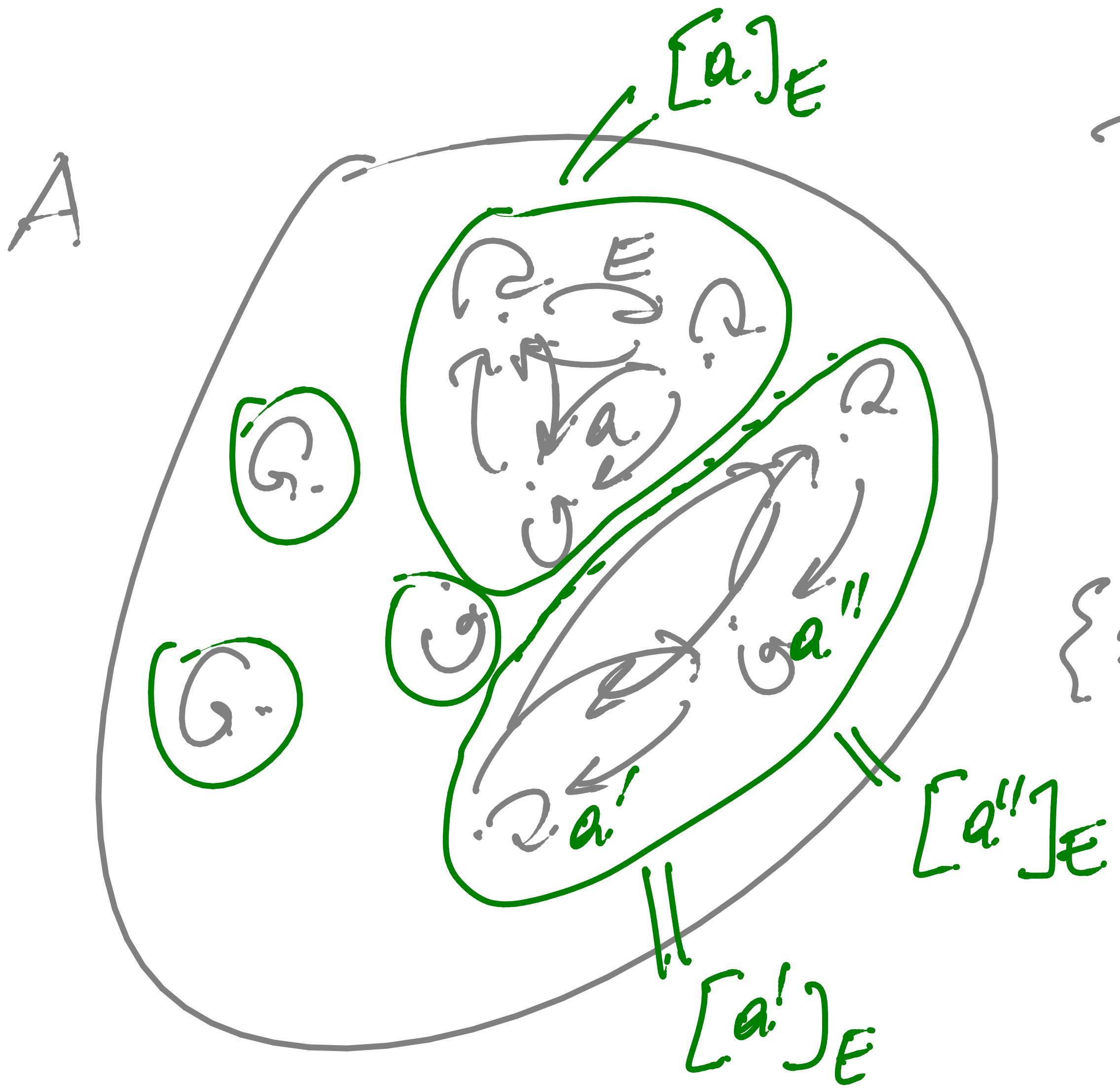
(1) Define a mapping that to every equivalence relation $E \subseteq A \times A$ associates a partition $\pi(E)$ of A.

[$\pi(E)$ is typically denoted $A/_E$ and referred to as the quotient of A under E]

Define $A/_E = \{ \underbrace{[a]_E}^{\subseteq A} \mid a \in A \}$

$= \{ S \subseteq A \mid \exists\, a \in A .\ S = [a]_E \}$

formally



For every $a \in A$

$$[a]_E \subseteq A$$

$\overset{def}{=}$ $\{x \in A \mid a \, E \, x\}$

Lemma: $a' E a'' \iff [a']_E = [a'']_E$

Lemma: $a \in [a]_E$

$a E a' \overset{?}{\Rightarrow} [a]_E = [a']_E$

Assume $\underline{a E a'}$ and let $x \in [a]_E$ $\rightsquigarrow [a]_E \subseteq [a']_E$

$\underline{RTP}$: $x \in [a']_E$

$\Updownarrow$?

$a' E x$

$a E x$

$a' E a$

$\Downarrow$

$a' E x$ ✓

$\boxtimes$

$[a]_E = [a']_E \overset{?}{\Longrightarrow} a E a'$

Assume $[a]_E = [a']_E$ Since $a \in [a]_E = [a']_E \Rightarrow a E a'$ ✓ $\boxtimes$

We have define

$$E \subseteq A \times A \longmapsto A/_E = \{ [a]_E \mid a \in A \}$$

but for it to be a function

$$\mathrm{EqRel}(A) \longrightarrow \mathrm{Part}(A)$$

we need check that whenever E is an equivalence relation $A/_E$ is a partition!

That is:

- each block in A/E is non-empty. ✓
- for all blocks $[a]_E \neq [a']_E$

$$\Downarrow ?$$

$$[a]_E \cap [a']_E = \emptyset$$

$$[a]_E \cap [a'] \neq \emptyset \implies \exists x.\ x \in [a]_E \wedge x \in [a']_E$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x \mathrel{E} a \quad : \quad x \mathrel{E} a'$$

$\vdots$

<u>exercise</u>

- $\bigcup \{ [a]_E \subseteq A \mid a \in A \} = A$

$(\subseteq)$ ✓

$(\supseteq)$ because $a \in [a]_E$.

$\square$

We have

$$\mathrm{Eq.Rel}(A) \longrightarrow \mathrm{Part}(A).$$
$$E \xmapsto{\pi} A/E$$

Let us define

$$\mathrm{Part}(A) \xrightarrow{\mathcal{E}} \mathrm{EqRel}(A)$$

Given $P \in \underline{\mathrm{Part}}(A)$, let $\mathcal{E}(P) \subseteq A \times A$ be defined by

$$x \; \mathcal{E}(P) \; y \overset{\text{def}}{\iff} \exists\, b \in P.\; x \in b \wedge y \in b$$

for $x, y \in A$.

To be sure that we have a function

$$\mathcal{E} : \underline{\mathrm{Part}}(A) \longrightarrow \underline{\mathrm{Eq.Rel}}(A)$$

we need check that for all $P \in \underline{\mathrm{Part}}(A)$ the relation $\mathcal{E}(P)$ is in fact an equiv. relation; that is,

$$\forall a \quad a \; \mathcal{E}(P) \; a$$

$$\forall a, a' \quad a \; \mathcal{E}(P) \; a' \Rightarrow a' \; \mathcal{E}(P) \; a$$

$- \forall a\, a'\, a''.\ a\, \mathcal{E}(P)\, a' \wedge a'\, \mathcal{E}(P)\, a'' \Rightarrow a\, \mathcal{E}(P)\, a''.$

We have

$$\begin{tikzcd} \mathrm{Part}(A) \arrow[rr, bend left, "\mathcal{E}"] & & \mathrm{EqRel}(A) \arrow[ll, bend left, "\pi"] \end{tikzcd}$$

Need check that

$\forall P \in \underline{\mathrm{Part}}(A).\quad \pi(\mathcal{E}(P)) = P$ exercise.

$\wedge$

$\forall E \in \underline{\mathrm{EqRel}}(A)\quad \mathcal{E}(\pi(E)) = E\quad ?$

$$\mathcal{E}(\pi(E)) = E$$

iff $\forall x,y \in A.\quad \underbrace{x \; \mathcal{E}(\pi(E)) \; y}_{} \iff x \, E \, y$

$\Updownarrow$ by def

$(\Longrightarrow)$

$\exists a \in A \;\exists\; [a]_E \in \pi(E) \,.\quad x \in [a]_E \wedge y \in [a]_E$

$\Updownarrow$ by def $\qquad \Updownarrow$ by def

$$\frac{x E a \qquad y E a}{x E y}$$

$(\Longleftarrow)$ . . . exercise.

Calculus of bijections

- $A \cong A$, $A \cong B \implies B \cong A$, $(A \cong B \wedge B \cong C) \implies A \cong C$
- If $A \cong X$ and $B \cong Y$ then

$$\mathcal{P}(A) \cong \mathcal{P}(X) \quad , \quad A \times B \cong X \times Y \quad , \quad A \uplus B \cong X \uplus Y \quad ,$$

$$\mathrm{Rel}(A, B) \cong \mathrm{Rel}(X, Y) \quad , \quad (A \rightharpoonup B) \cong (X \rightharpoonup Y) \quad ,$$

intuition

$$(A \Rightarrow B) \cong (X \Rightarrow Y) \quad , \quad \mathrm{Bij}(A, B) \cong \mathrm{Bij}(X, Y)$$

In algebraic form

$$a = x \wedge b = y \Rightarrow b^a = y^x$$

- $A \cong [1] \times A$, $(A \times B) \times C \cong A \times (B \times C)$, $A \times B \cong B \times A$
- $[0] \uplus A \cong A$, $(A \uplus B) \uplus C \cong A \uplus (B \uplus C)$, $A \uplus B \cong B \uplus A$
- $[0] \times A \cong [0]$, $(A \uplus B) \times C \cong (A \times C) \uplus (B \times C)$
- $(A \Rightarrow [1]) \cong [1]$, $\boxed{(A \Rightarrow (B \times C)) \cong (A \Rightarrow B) \times (A \Rightarrow C)}$
- $([0] \Rightarrow A) \cong [1]$, $((A \uplus B) \Rightarrow C) \cong (A \Rightarrow C) \times (B \Rightarrow C)$
- $([1] \Rightarrow A) \cong A$, $\boxed{((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))}$
- $(A \rightharpoonup B) \cong (A \Rightarrow (B \uplus [1]))$
- $\boxed{\mathcal{P}(A) \cong (A \Rightarrow [2])}$

$c^{a \cdot b} = (c^a)^b$

$(b \cdot c)^a = b^a \cdot c^a$

$\# \mathcal{P}(A) = 2^{\# A}$ (A finite)