Bijections

A function $f : A \to B$ is a bijection whenever it has a (two-sided) inverse; that is, there is $g : B \to A$ such that

$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_B.$$

NB: Inverses, if they exist, are unique. Typically, we write f^{-1} for the inverse of f.

$$f^{-1} \circ f = id_A \qquad f \circ f^{-1} = id_B.$$

Recall

$$f \circ g = \mathrm{id}_B$$

$$\iff \forall b \in B.\ f(g(b)) = b$$

$$\wedge$$

$$g \circ f = \mathrm{id}_A$$

$$\iff \forall a \in A.\ g(f(a)) = a$$

Bijections

Definition 127 *A function* $f : A \to B$ *is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function* $g : B \to A$ *(referred to as the* inverse *of* f*) such that*

1. g *is a* retraction *(or* left inverse*) for* f*:*

$$g \circ f = \mathrm{id}_A \quad ,$$

2. g *is a* section *(or* right inverse*) for* f*:*

$$f \circ g = \mathrm{id}_B \quad .$$

Proposition 129 *For all finite sets* A *and* B*,*

$$\#\,\mathrm{Bij}(A,B) \;=\; \begin{cases} 0 & \text{, if } \#A \neq \#B \\ n! & \text{, if } \#A = \#B = n \end{cases}$$

PROOF IDEA:

$A = \{a_1, \ldots, a_m\}$ $\quad$ $B = \{b_1, \ldots, b_n\}$

$\underline{n < m}$

$\Downarrow$

there cannot be a bijection

$$\begin{array}{cccc} a_1 & a_2 & \ldots & a_m \\ \downarrow & \nearrow\!\!\swarrow & & \\ b_{i_1} & & & \end{array}$$

suppose $\Longrightarrow$ NO BIJECTION

we have less b's than a's.

if $n < m$ then for every function $f: A \to B$ we have the situation

$$\begin{tikzcd} a_i \arrow[dr, "f"] & & a_j \arrow[dl, "f"'] \\ & b_k & \end{tikzcd}$$

$\Rightarrow$ we have less a's than b's

if $m < n$ then for every function $f: A \to B$

$$\begin{tikzcd} a_1 \arrow[d, "f"] & \cdots \arrow[d] & a_m \arrow[drrrr, bend left] & & & & \\ b_1 & \cdots & \cdots & b_m & b_{m+1} & \cdots & b_n \end{tikzcd}$$

There is some k such that $b_k \neq f(a_i)$ for all $i = 1, \ldots, m$ and so cannot construct an inverse (in particular we don't where the inverse should map the b_k into to some a).

We have a bijection precisely when $m = n$.

$A = \{a_1 \text{———} a_m\}$ $B = \{b_1 \text{———} b_m\}$

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_m \\ \downarrow & \downarrow & & \downarrow \\ b_{i_1} & b_{i_2} & \cdots & b_{i_m} \end{array} \qquad \begin{array}{c} a_j \\ \uparrow \text{ inverse} \\ b_{i_j} \end{array}$$

$\{i_1, i_2, \ldots, i_m\} = \{1, \ldots, m\}$

The i_k's are a permutation of $1, \ldots, m$

and there are $m!$ of those.

Theorem 130 *The identity function is a bijection, and the composition of bijections yields a bijection.*

$\underline{\mathrm{Bij}}(A,B) \subseteq (A \Rightarrow B) \subseteq (A \Rightarrow B) \subseteq \mathrm{Rel}(A,B)$

id_A bijection

$A \xrightarrow{f} B \xrightarrow{g} C$ bijections $\Rightarrow$ $g \circ f : A \to C$ bijection.

Definition 131 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

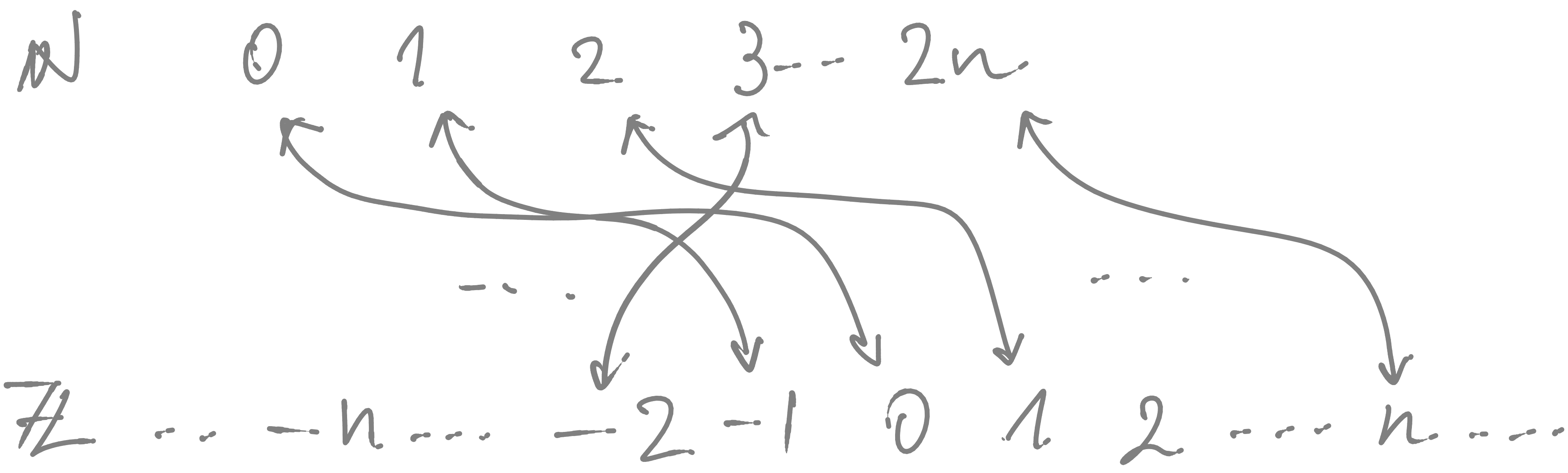
2. $\mathbb{N} \cong \mathbb{N}^{+}$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.

$\quad\quad\; \shortparallel$

$\quad\quad\; \{n \in \mathbb{N} \mid n > 0\}$

$$\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "+1"] & \mathbb{N}^+ \arrow[l, bend left, "-1"] \end{tikzcd} \implies \mathbb{N} \cong \mathbb{N}^+$$

$$\begin{array}{llllllllll}
\mathbb{N} & 0 & 1 & 2 & 3 & \cdots & n & \cdots & & \\
 & \updownarrow & \updownarrow & \updownarrow & & & & \searrow & & \\
\mathbb{N}^+ & 1 & 2 & 3 & & \cdots & n & n+1 & \cdots &
\end{array}$$



$$\begin{tikzcd} \mathbb{N} \arrow[loop left, "\mathrm{id}_{\mathbb{N}}"] \arrow[r, bend left] & \mathbb{Z} \arrow[l, bend left] \arrow[loop right, "\mathrm{id}_{\mathbb{Z}}"] \end{tikzcd} .$$

exercise

$\mathbb{N}\times\mathbb{N}$

$$
\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & \cdots & \cdots & m & \cdots \\
\hline
0 & 0 & 2 & 3 & 9 & 10 & & \\
1 & 1 & 4 & 8 & 11 & & & \\
2 & 5 & 7 & 12 & & & & \\
\vdots & 6 & 13 & & & & & \\
n & 14 & & & & & & \\
\vdots & & & & & & &
\end{array}
$$

(n,m)

exercise

$$
\begin{tikzcd}
\mathbb{N}\times\mathbb{N} \arrow[r, bend left] & \mathbb{N} \arrow[l, bend left]
\end{tikzcd}
$$

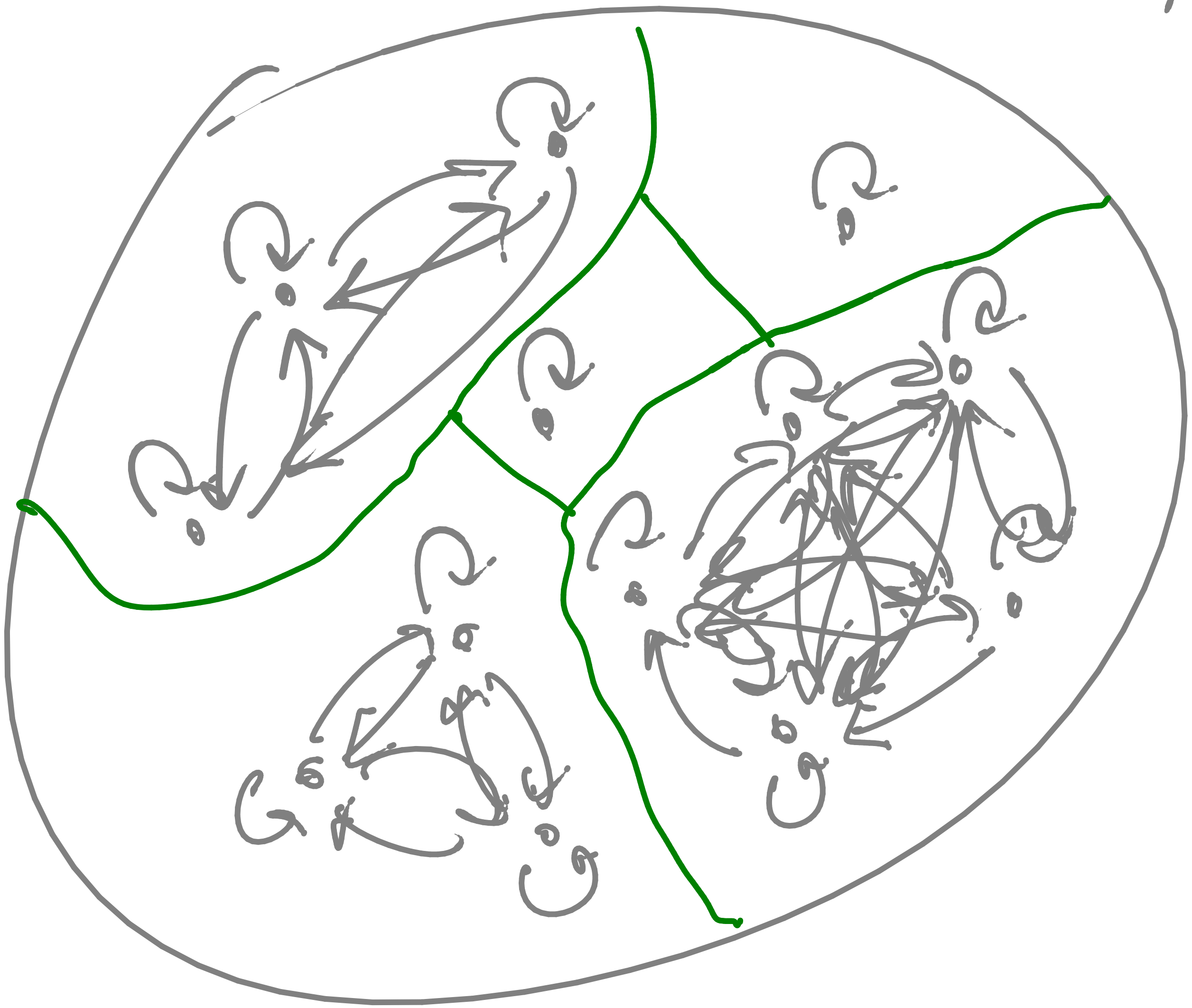
Equivalence relations and set partitions

- Equivalence relations. Fix a set A.

$R \subseteq A \times A$ is an equivalence relation.

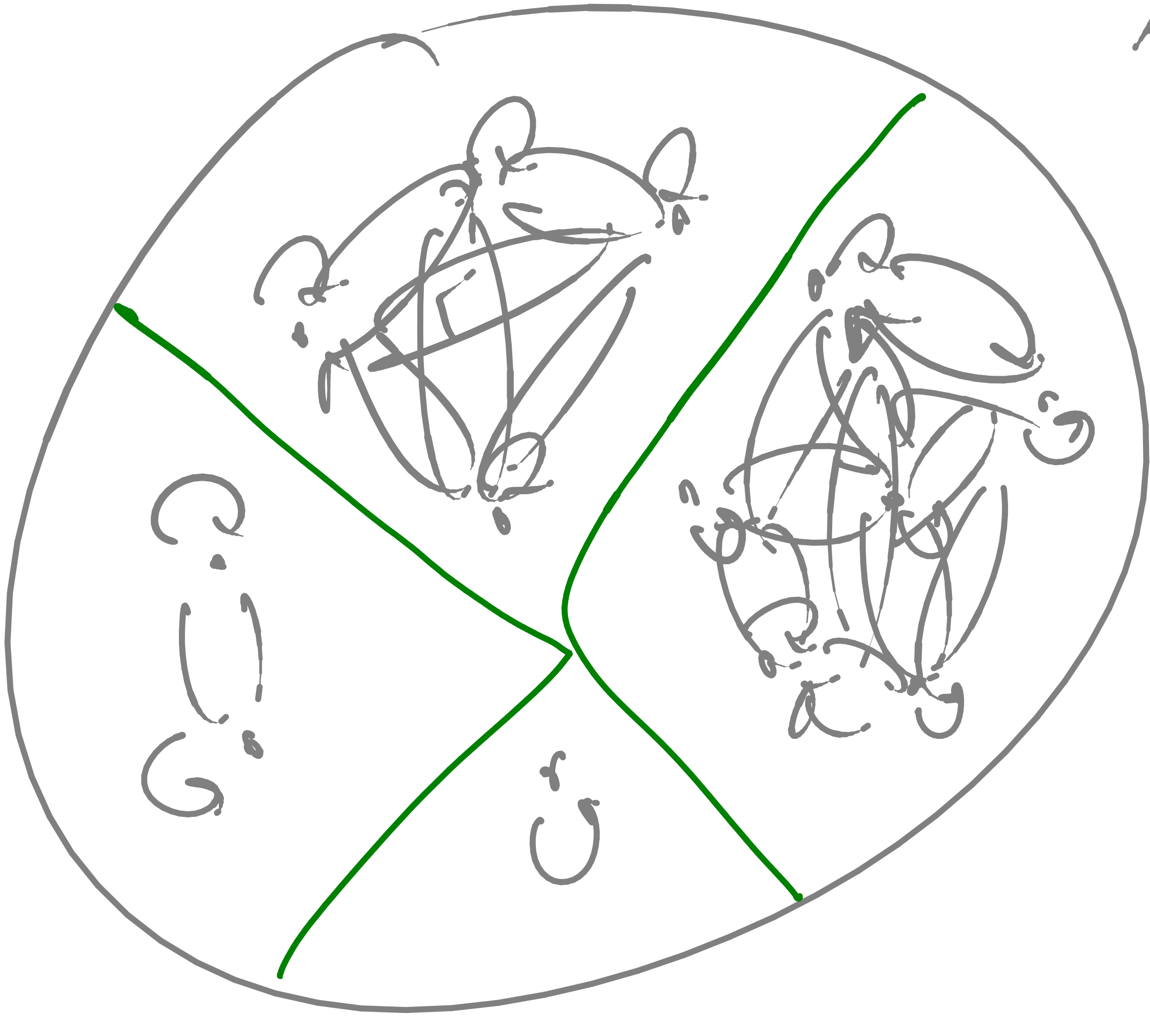
- <u>reflexivity</u> $\forall a \in A.\ a R a$
- <u>symmetry</u> $\forall a, a' \in A.\ a R a' \Rightarrow a' R a$
- <u>transitivity</u> $\forall a, a', a'' \in A.$
  $a R a' \wedge a' R a'' \Rightarrow a R a''$

$\underline{EqRel}(A) = \{ R \subseteq A \times A \mid R \text{ is an equiv. rel.} \}$



A

A



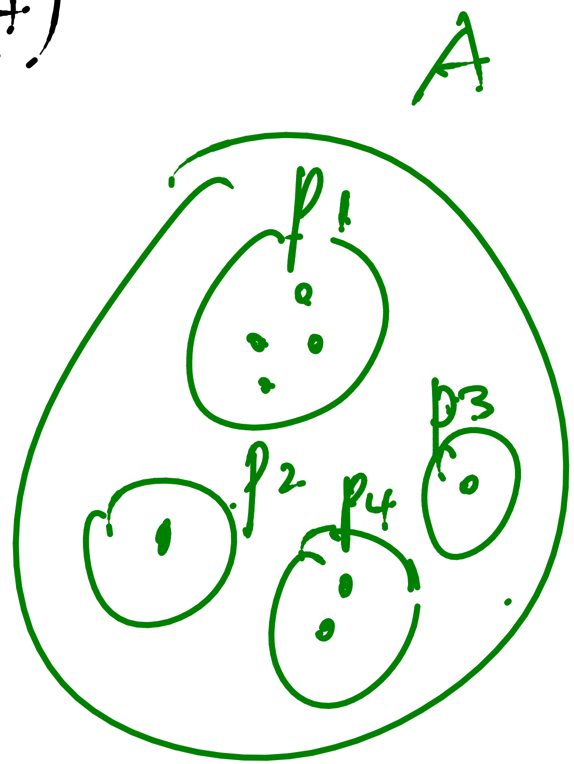
$\underline{\mathrm{Part}}(A) = \{ \pi \mid \pi \text{ is a partition of } A \}$

<u>Idea</u>:

$$\underline{\mathrm{EqRel}}(A) \cong \mathrm{Part}(A)$$

$\pi \subseteq \mathcal{P}(A)$

- $\forall p \in \pi .\ p \neq \emptyset$
- $\forall p, p' \in \pi .\ p \neq p' \Rightarrow p \cap p' = \emptyset$
- $\bigcup \pi = A$



$\pi = \{ p_1, p_2, p_3, p_4 \}$

- Set partitions.