Partial functions

Definition 119 *A relation* $R : A \rightarrow\!\!\!\!\!+\; B$ *is said to be* functional, *and called a* partial function, *whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B.\ a\,R\,b_1 \wedge a\,R\,b_2 \implies b_1 = b_2 \quad .$$

$$\begin{array}{ccc} & & b_1 \\ & \nearrow & \\ a & R & \\ & \searrow & \\ & & b_2 \end{array} \implies b_1 = b_2$$

Example

$\emptyset : A \rightarrow\!\!\!\!\!+\; B$ is a partial function.

If $f \subseteq A \times B$ is a partial function

we write, for $a \in A$,

$f(a)\downarrow$ if $\exists b \in B.\ a \, f \, b$ — there is an "output" on a

$f(a)\uparrow$ if $\forall b \in B.\ \neg(a \, f \, b)$ — There is no "output" on a

<u>Notation</u> For partial functions $f : A \rightharpoonup B$

we write, for $a \in A$ such that $f(a)\downarrow$, for

the $b \in B$ with $a \, f \, b$

Theorem 121 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

NB

$$f = g : A \rightharpoonup B$$

notation for partial functions.

iff

$$\forall a \in A. \, \big(f(a)\!\downarrow \iff g(a)\!\downarrow \big) \wedge f(a) = g(a)$$

f and g are defined (or give an "output") on a.

Notation:

To define a partial function

$$f : A \to B$$

we typically write

$$a \overset{f}{\longmapsto} \text{---}\, a \,\text{---}$$

mapping

↳ "expression"

and understand that

$$f = \{ (a, \text{---}\, a \,\text{---}) \mid a \in A \}$$

and check that it has the property of partial function

$f: A \to B$ its domain of definition, $\underline{\mathrm{dom}}(f)$, is $\{ a \in A \mid f(a) \downarrow \}$

Example: The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{N}$:

$\| \quad \{ a \in A \mid \exists b .\, a f b \}$

- for $n \geq 0$ and $m > 0$,
  $(n, m) \mapsto (\, \mathrm{quo}(n, m)\, ,\ \mathrm{rem}(n, m)\,)$
- for $n \geq 0$ and $m < 0$,
  $(n, m) \mapsto (\, -\mathrm{quo}(n, -m)\, ,\ \mathrm{rem}(n, -m)\,)$
- for $n < 0$ and $m > 0$,
  $(n, m) \mapsto (\, -\mathrm{quo}(-n, m) - 1\, ,\ \mathrm{rem}(m - \mathrm{rem}(-n, m), m)\,)$
- for $n < 0$ and $m < 0$,
  $(n, m) \mapsto (\, \mathrm{quo}(-n, -m) + 1\, ,\ \mathrm{rem}(-m - \mathrm{rem}(-n, -m), -m)\,)$

Its domain of definition is $\{\, (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \,\}$.

The number of relations between finite sets.

$$\#A = n \qquad \#B = m$$

$$\#\underline{Rel}(A,B) = \#\mathcal{P}(A \times B)$$

$$= 2^{\#(A\times B)}$$

$$= 2^{\#A \cdot \#B}$$

$$\#\mathcal{P}(X) = 2^{\#X}$$

$$\#(A\times B) = \#A\,\#B$$

$(A \rightharpoonup B) = \{ f \in \mathrm{Rel}(A,B) \mid f \text{ is a partial function} \}$

Proposition 122 *For all finite sets* A *and* B,

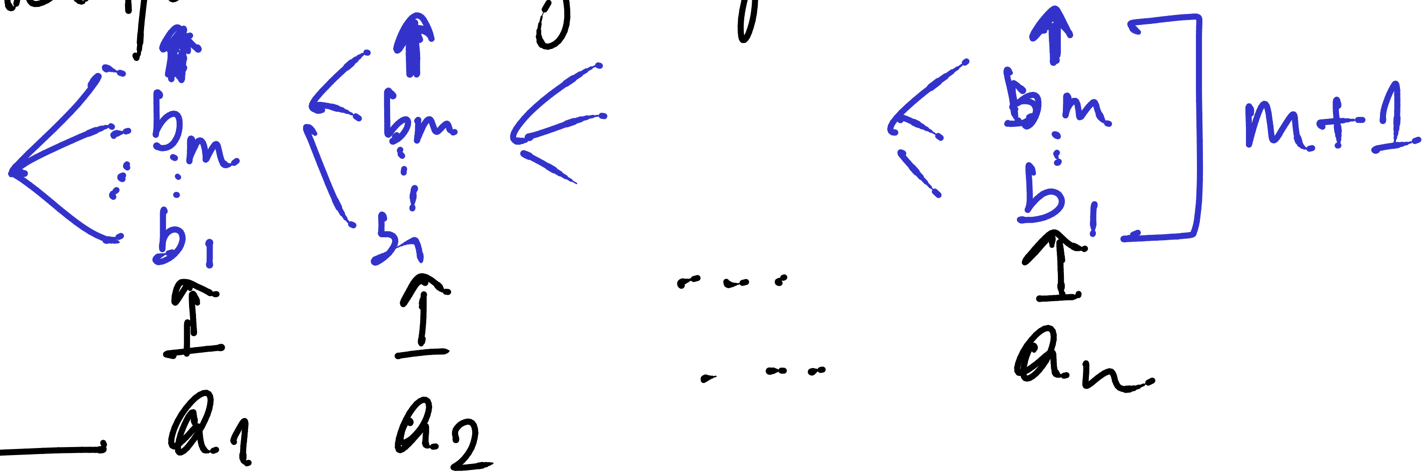
$$\#(A \rightharpoonup B) = (\#B + 1)^{\#A} \quad .$$

PROOF IDEA:

$\#A = n \qquad A = \{a_1, \ldots, a_n\}$

$\#B = m \qquad B = \{b_1, \ldots, b_m\}$

A partial function may be presented as:



$= \{ f \subseteq A \times B \mid f \text{ is a partial function} \}$

Functions (or maps)

the subclass of partial functions for which every "input" has an "output".

Definition 123 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

$f : A \to B$ total

$\iff$ for all $a \in A.\ \exists b \in B.\ a f b$.

$a \overset{f}{\longmapsto} f(a)$

$(A \Rightarrow B)$ is the set of all functions from A to B.

Theorem 124 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ a\, f\, b \quad .$$

$A \overset{\vert}{\rightarrow} C$ is it a partial function where
$g \circ f$ f and g are?

NB:

$$(A \Rightarrow B) \subseteq (A \rightharpoonup B) \subseteq \underline{\mathrm{Rel}}(A, B)$$

$$\underline{\mathrm{Rel}}(A, B) = \mathcal{P}(A \times B)$$

$id_A : A \to A$ $\qquad$ $id_A : A \rightharpoonup A$ $\qquad$ $id_A : A \overset{\vert}{\rightarrow} A$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, harpoon, "f"] \arrow[rr, harpoon, bend right, "g \circ f"'] & B \arrow[r, harpoon, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "R", "\vert" marking] \arrow[rr, bend right, "S \circ R"', "\vert" marking] & B \arrow[r, "S", "\vert" marking] & C \end{tikzcd}$$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

a'

$a \mapsto b = f(a)$

b'

$b'' \mapsto c = g(b'')$

$(g \circ f)(a)\downarrow \iff f(a)\downarrow$

$\wedge\ g(f(a))\downarrow$

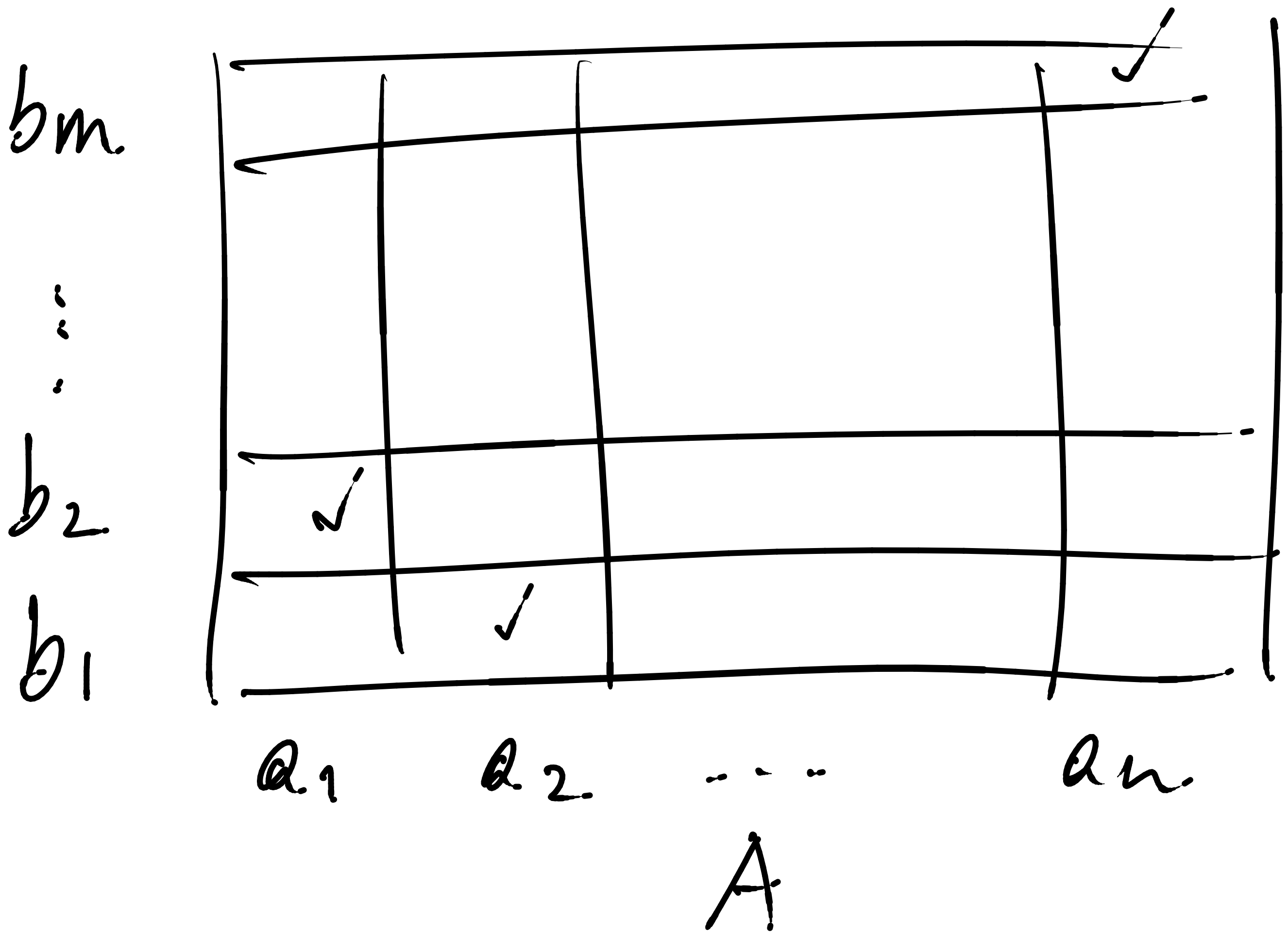
$\wedge\ (g \circ f)(a) = g(f(a))$

<u>Exercise</u>: $(g \circ f)(a) = g(f(a))$ whenever $f(a)\downarrow$

Proposition 125 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) = \#B^{\#A} \quad .$$

PROOF IDEA: e.g. $\{a_1 \mapsto b_2, a_2 \mapsto b_1, \ldots, a_n \mapsto b_m\}$



Theorem 126 *The identity partial function is a function, and the composition of functions yields a function.*

NB

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

$$\mathrm{id}_A(a) = a$$

and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

$$\big(g \circ f\big)(a) = g\big(f(a)\big) \quad .$$