$$\| \; \{0, \ldots, m-1\}$$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

$$\begin{tikzcd}
{[m]} \arrow[r, "R", "|" marking] \arrow[rr, bend right, "|" marking, "S \circ R"'] & {[n]} \arrow[r, "S", "|" marking] & {[\ell]}
\end{tikzcd}$$

$$S \circ R$$

$\|$ <u>exercise</u>

$$\underline{rel}\left(\underline{mat}(R) \cdot \underline{mat}(S)\right)$$

$\underline{mat}(R)$ $(m \times n)$-matrix

$\underline{mat}(S)$ $(n \times \ell)$-matrix

$\underline{mat}(R) \cdot \underline{mat}(S)$ $(m \times \ell)$-matrix

$$R \subseteq [3] \times [2]$$

$$R = \{ (1,1), (2,1), (0,1) \}$$

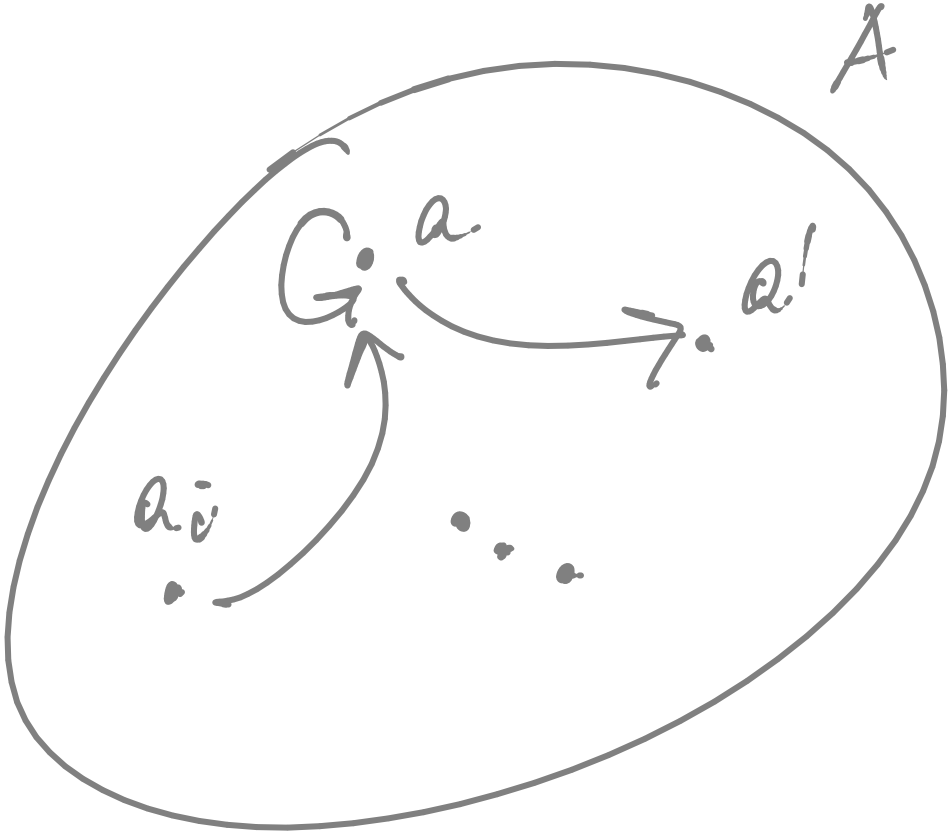
$$\underline{mat}(R) = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \\ \hline 0 & 1 \end{array} \right] \end{array}$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \underline{rel}(M) = \{ (0,0), (1,1), (2,0) \}$$

Directed graphs

$R \subseteq A \times A$

Definition 108 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



$(a, a) \in R$

$(a, a') \in R$

$(a_i, a) \in R$

$\underline{\mathrm{Rel}}(A) = \underline{\mathrm{Rel}}(A, A)$ = the set of directed graphs on A

Corollary 110 *For every set A, the structure*

$$\left(\ \mathrm{Rel}(A)\ ,\ \mathrm{id}_A\ , \circ\ \right)$$

is a monoid.

Definition 111 *For $R \in \mathrm{Rel}(A)$ and $n \in \mathbb{N}$, we let*

$$R^{\circ n} = \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \in \mathrm{Rel}(A)$$

be defined as id_A for $n = 0$, and as $R \circ R^{\circ m}$ for $n = m + 1$.

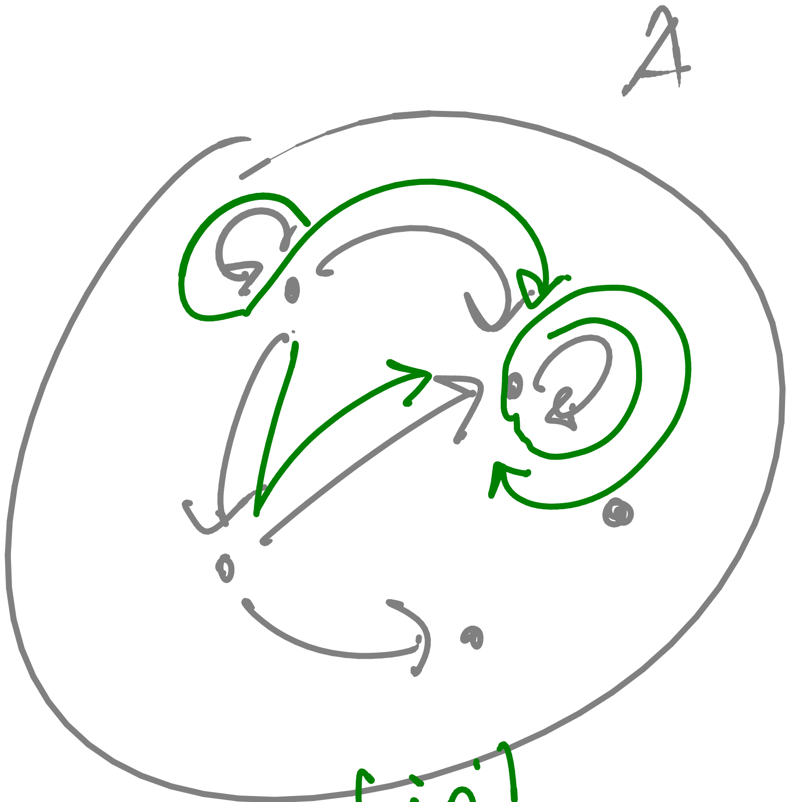
$R^{\circ n}$ is the relation that tracks the paths of length n in the directed graph given R.

Paths

Proposition 113 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s \; R^{\circ n} \; t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF:

paths of length 2



$R^{\circ 2} = R \circ R =$ has the pairs (i,j) of the directed graph for which there is a path of length 2 from i to j

Proceed by induction on n ext.

<u>Base case</u> <u>RTP</u> $\underbrace{s\, R^{0}}_{\text{def id}}\, t$ iff $\exists$ a path of length 0 from s to t

$\Updownarrow$ def

$s = t$

✓

Inductive step

$s\, R^{\circ(n+1)}\, t$ iff $\exists$ path of lenght $n+1$ from s to t

$\Updownarrow$ def

$s\, (R \circ R^{\circ n})\, t$

$\Updownarrow$

$\exists u.\ s R u \wedge u R^{\circ n} t$

$\Updownarrow$ by induction

$\exists$ a path of length n from u to t.

$$R^{\circ *} = \bigcup \{ \mathrm{id}_A, R, R\circ R, \ldots, \underbrace{R\circ \cdots \circ R}_{n}, \ldots \}$$

$$= \mathrm{id}_A \cup R \cup (R\circ R) \cup \cdots \cup (\underbrace{R\circ \cdots \circ R}_{n}) \cup \cdots$$

Definition 114 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 115 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R.

In terms of matrices

$$M^* = I + M + \cdots + M^n + \cdots$$

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 &= I_n \\ M_{k+1} &= I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Preorders

Definition 116 *A* <u>preorder</u> $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq\, \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

P

$$\forall x, y, z \in P.\ (x \sqsubseteq y \wedge y \sqsubseteq z) \implies x \sqsubseteq z$$

$x_0 \sqsubseteq x_1 \wedge x_1 \sqsubseteq x_2 \wedge \cdots \wedge x_{n-1} \sqsubseteq x_n$

$\Downarrow$

$x_0 \sqsubseteq x_2$ $\Downarrow$ $x_0 \sqsubseteq x_3$ $\Downarrow$ $\cdots$ $x_0 \sqsubseteq x_n$

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\mathbb{Z}, \mid)$.

partial orders (or posets)
$\overset{\text{def}}{=}$ preorders
s.t.:
(antisymmetry)
$x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

a preorder that is not a partial order.

Theorem 118 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid \overbrace{R \subseteq Q \wedge Q \textit{ is a preorder}}^{\text{closure property.}} \} \quad .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF: $\mathcal{F}_R$ is the family of all the preorders on A that contain R.

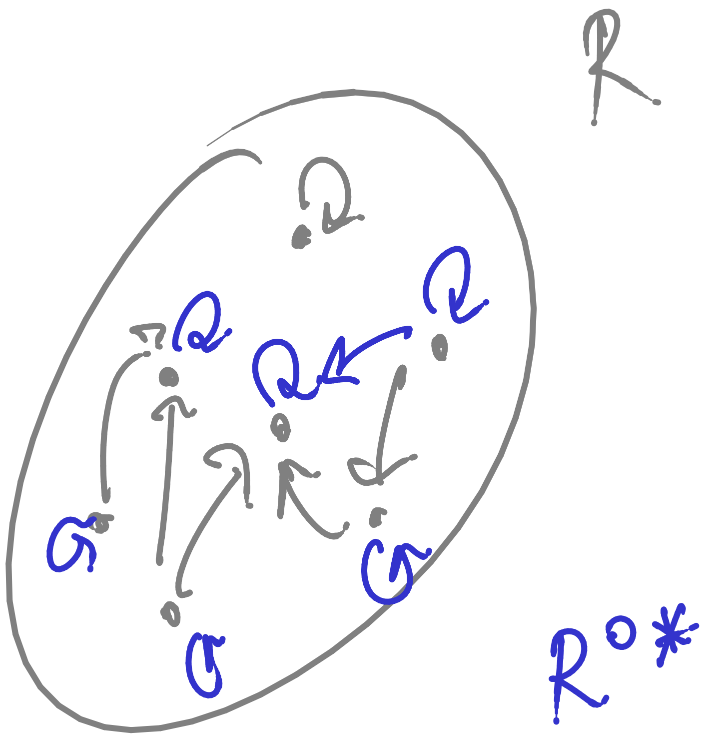
$\bigcap \mathcal{F}_R \subseteq R^{\circ *}$

Show

$R^{\circ *} \subseteq Q$

$\forall Q$ s.t. $R \subseteq Q$

$\wedge$ Q preorder.



(i) $R^{\circ *} \in \mathcal{F}_R$

- $R \subseteq R^{\circ *} = \bigcup_{n \in \mathbb{N}} R^{\circ n}$

 $\underline{\text{NB}}$: $R^{\circ 1} = R$

- $R^{\circ *}$ is a preorder:
  - $x R^{\circ *} x \ \forall x$ because $R^{\circ 0} = \mathrm{id}$
  - $x R^{\circ *} y \wedge y R^{\circ *} z \overset{?}{\Longrightarrow} x R^{\circ *} z$

uses the characterization of $R^{\circ *}$ as describing paths in R.