A relation from A to B is a set consisting of pairs with first component in A and second component in B.

Relations

Definition 99 *A* (binary) relation R *from a set* A *to a set* B

$$R : A \longrightarrow\!\!\!\!\!+ \; B \quad \textit{or} \quad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \quad \textit{or} \quad R \in \mathcal{P}(A \times B) \quad .$$

Notation 100 *One typically writes* $a\,R\,b$ *for* $(a, b) \in R$.

Examples:

- Empty relation.
  $\emptyset : A \nrightarrow B$ $\qquad$ $(a\ \emptyset\ b \iff \mathbf{false})$

- Full relation.
  $(A \times B) : A \nrightarrow B$ $\qquad$ $(a\ (A \times B)\ b \iff \mathbf{true})$

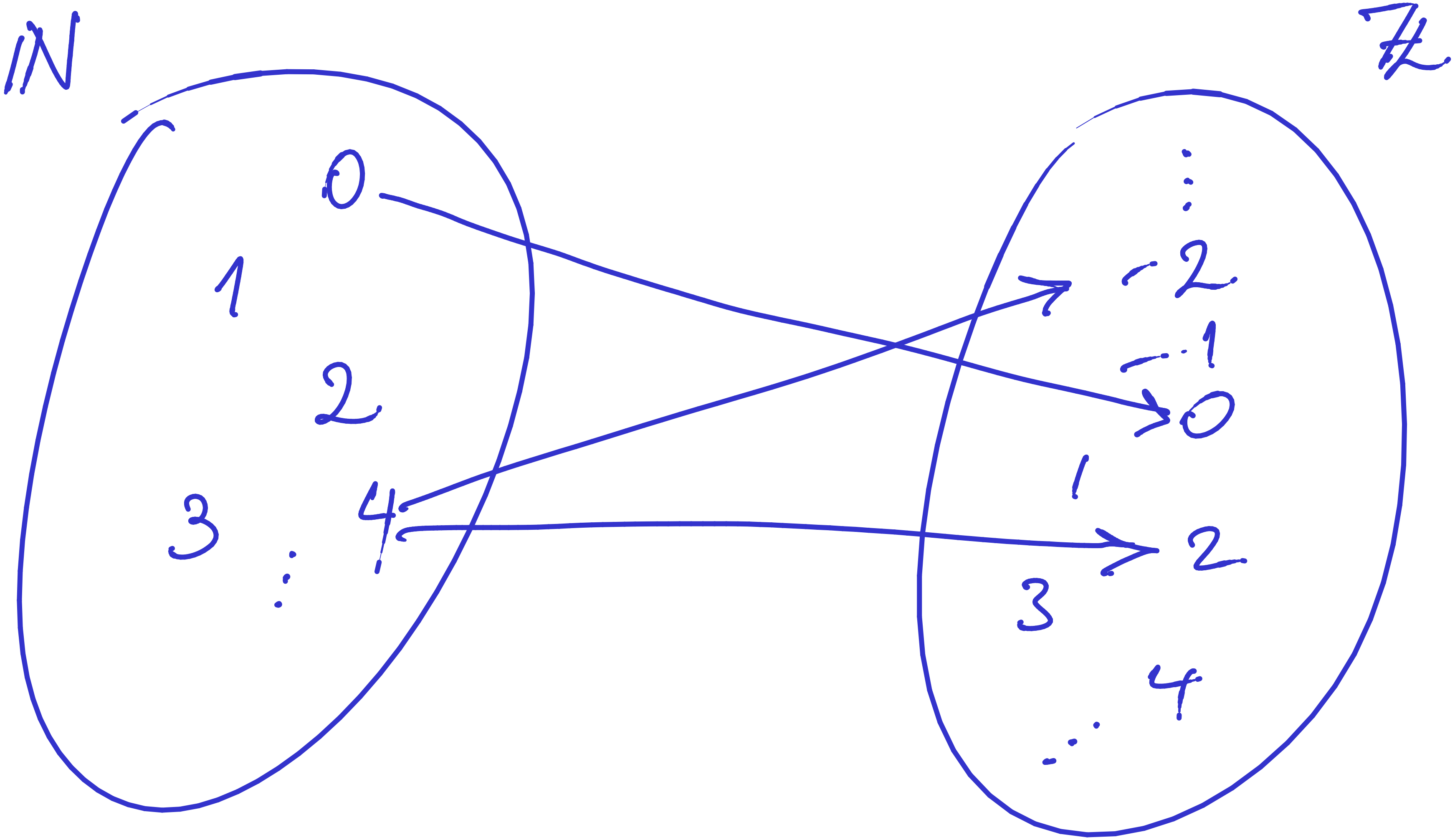
- Identity (or equality) relation.
  $\mathrm{id}_A = \{\ (a, a) \mid a \in A\ \} : A \nrightarrow A$ $\qquad$ $(a\ \mathrm{id}_A\ a' \iff a = a')$

- Integer square root.
  $R_2 = \{\ (m, n) \mid m = n^2\ \} : \mathbb{N} \nrightarrow \mathbb{Z}$ $\qquad$ $(m\ R_2\ n \iff m = n^2)$

$m R_2 n \iff m = n^2$

$(0,0) \in R_2$

$0 R_2 0$

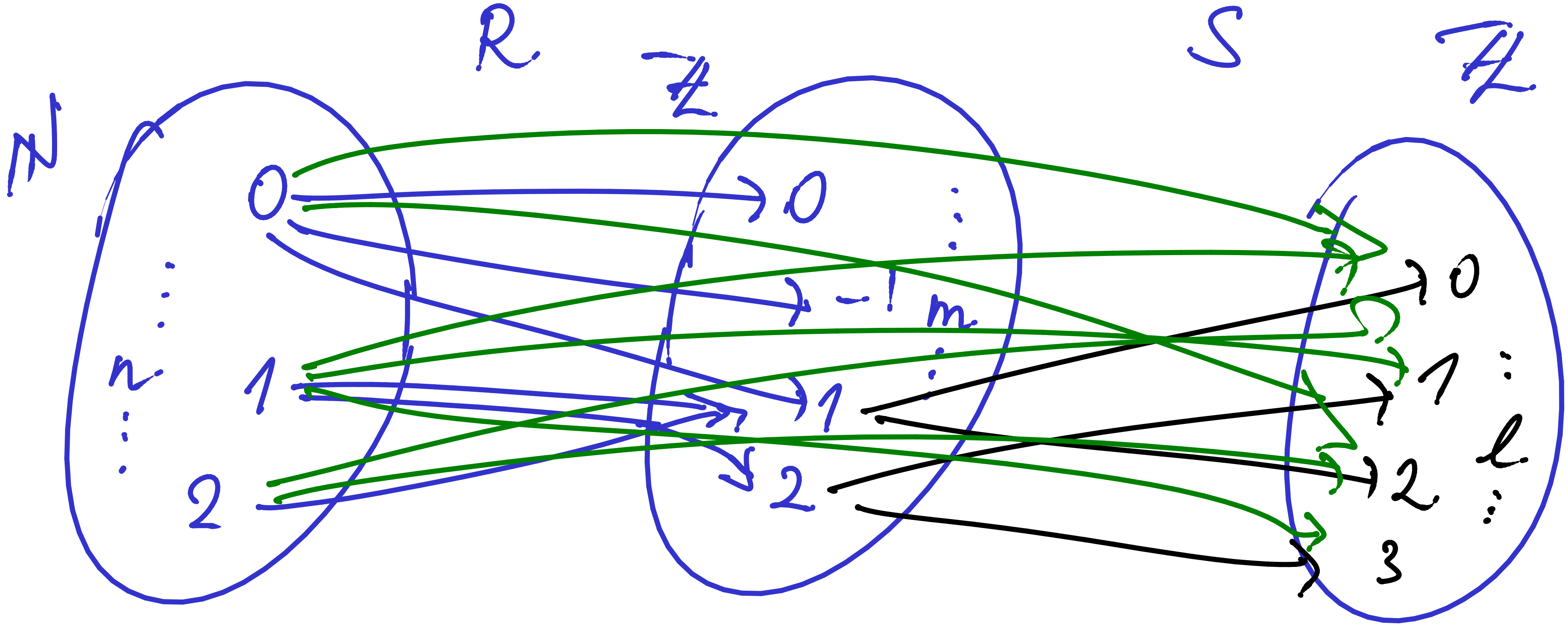


Internal diagrams

Example: $S \circ R : \mathbb{N} \rightarrow\!\!\!\!\!+\ \mathbb{Z}$

$$R = \{\ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1)\ \} : \mathbb{N} \rightarrow\!\!\!\!\!+\ \mathbb{Z}$$

$$S = \{\ (1,0), (1,2), (2,1), (2,3)\ \} : \mathbb{Z} \rightarrow\!\!\!\!\!+\ \mathbb{Z}$$



Relational extensionality

$$R = S : A \rightarrow\!\!\!\!\!\!+\!\!\!\rightarrow B$$

iff

$$\forall a \in A.\, \forall\, b \in B.\ a\, R\, b \iff a\, S\, b$$

Relational composition

$$\frac{R : A \nrightarrow B \qquad S : B \nrightarrow C}{S \circ R : A \nrightarrow C}$$

$$R\,;S : A \nrightarrow C$$

$\forall a \in A\ \forall c \in C.$

$$a\,(S \circ R)\,c \overset{\text{df}}{\iff} \exists b \in B.\ aRb \wedge bSc$$

$$(S \circ R) = \{(a,c) \in A \times C \mid \exists b \in B.\ aRb \wedge bSc\}$$

Theorem 102 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

  For all $R : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$, $S : B \relbar\joinrel\mapstochar\joinrel\rightarrow C$, *and* $T : C \relbar\joinrel\mapstochar\joinrel\rightarrow D$,

  $$(T \circ S) \circ R \;=\; T \circ (S \circ R)$$

  $\rightsquigarrow$ wlog we can write:

  $T \circ S \circ R$

- *Neutral element.*

  For all $R : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$,

  $$R \circ \mathrm{id}_A \;=\; R \;=\; \mathrm{id}_B \circ R \quad .$$

$$(T\circ S)\circ R \overset{?\checkmark}{=} T\circ(S\circ R) : A \nrightarrow D$$

$$\iff$$

$\forall a\in A.\ \forall d\in D.$

$$a\,((T\circ S)\circ R)\,d \overset{?\checkmark}{\iff} a\,(T\circ(S\circ R))\,d$$

$$a\,((T\circ S)\circ R)\,d \overset{def}{\iff} \exists b\in B.\ aRb \wedge b\,(T\circ S)\,d$$

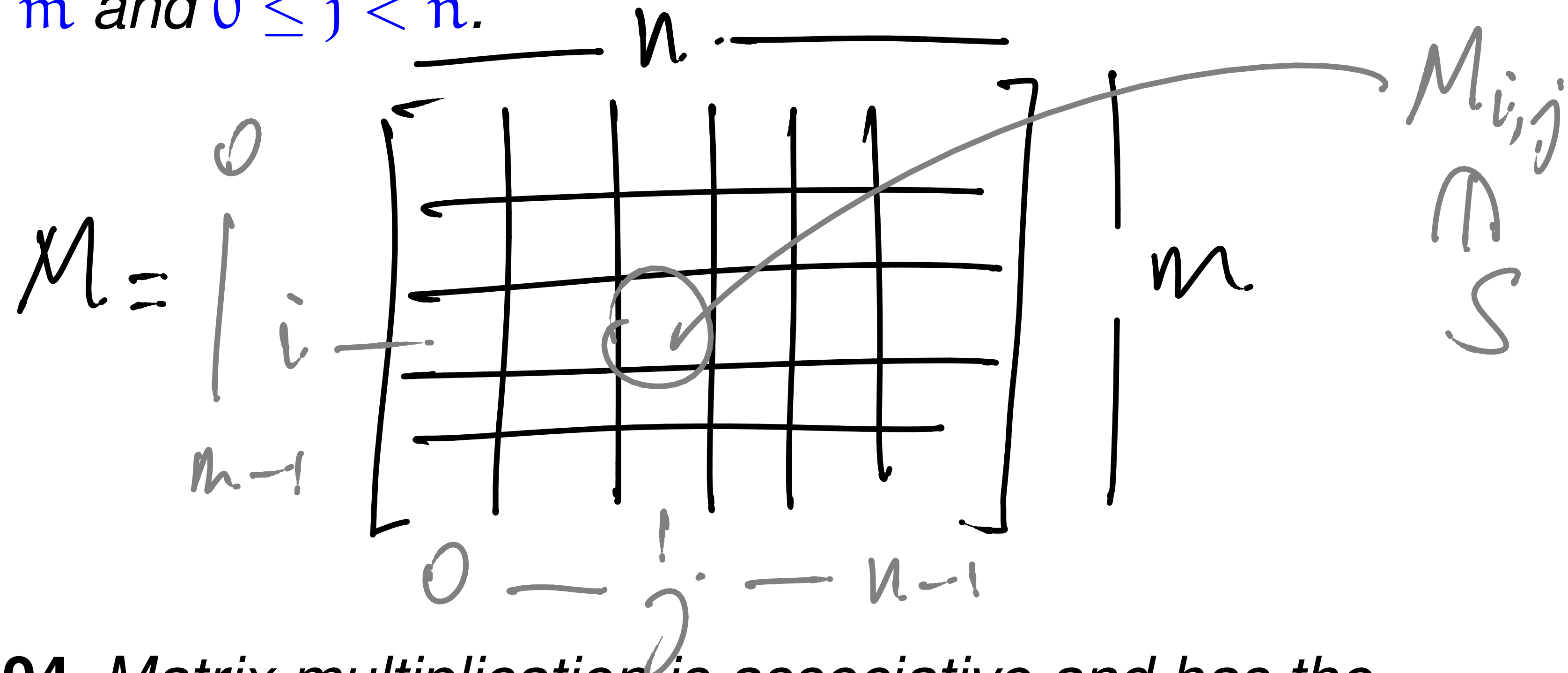
$$\overset{def}{\iff} \exists b\in B.\ aRb \wedge \exists c\in C.\ bSc \wedge cTd$$

$$a\,(T\circ(S\circ R))\,d \overset{def}{\iff} \exists z\in C.\ a\,(S\circ R)\,z \wedge zTd$$

$$\overset{def}{\iff} \exists z\in C.\ \exists y\in B.\ aRy \wedge ySz \wedge zTd.$$

Definition 103 $M, N \rightsquigarrow \left[\begin{array}{l} N+M \\ N \bullet M \end{array}\right.$

1. *For positive integers* m *and* n*, an* $(m \times n)$*-matrix* M *over a* $\boxed{\textit{semiring } (S, 0, \oplus, 1, \odot)}$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$.



Theorem 104 *Matrix multiplication is associative and has the identity matrix as neutral element.*

$$(N+M)_{i,j} \overset{def}{=} N_{i,j} \oplus M_{i,j}$$

$$(M \cdot N)_{i,j} = \sum_{k=0}^{n-1} m_{i,k} \odot n_{k,j}$$

$$\| $$

M $(m \times n)$-matrix

N $(n \times k)$-matrix

$$(m_{i,0} \odot n_{0,j}) \oplus \cdots \oplus (m_{i,n-1}\, n_{n-1,j})$$

$$a\,(S \circ R)\,c \iff \exists b.\; aRb \wedge bSc$$

Take the semiring $\mathbb{B}$ to be

$\{true, false\}$

with

$0 = false$, $\oplus = \vee$

$1 = true$, $\odot = \wedge$

What is a an $(m \times n)$-matrix over $\mathbb{B}$?

$M_{i,j} \in \{true, false\}$

$0 \leq i < m$

$0 \leq j < n$

$(m \times n)$-matrix over $\mathbb{B}$ $\rightsquigarrow$ $\underline{\mathrm{Rel}}([m],[n])$

where

$[k] \overset{\mathrm{def}}{=} \{0, \ldots, k-1\}$

$$M \longmapsto \underline{\mathrm{rel}}(M) \subseteq [m] \times [n]$$

$$\| $$

$$\{(i,j) \mid M_{i,j} = \mathrm{true}\}$$

$$\underline{\mathrm{Rel}}([m],[n]) \rightsquigarrow (m\times n)\text{-matrix}$$

$$R \subseteq [m]\times[n] \rightsquigarrow \underline{\mathrm{mat}}(R)$$

$$\underline{\mathrm{mat}}(R)_{i,j} \overset{\mathrm{def}}{=} \begin{cases} \mathrm{true}, & (i,j)\in R \\ \mathrm{false}, & (i,j)\notin R \end{cases}$$

$(m\times n)$-matrix

$$M \rightsquigarrow \underline{\mathrm{rel}}(M) \subseteq [m]\times[n]$$

$$\underline{\mathrm{mat}}(\underline{\mathrm{rel}}(M)) \overset{\text{exercise}}{=\!=} M$$

$R \subseteq [m] \times [n] \rightsquigarrow \underline{\text{mat}}(R)$ $(m \times n)$-matrix

$\underline{\text{rel}}(\underline{\text{mat}}(R))$

$\Vert$ exercise

R

These kind of processes are called bijections (or bijective correspondences).

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .