NB: We assume a pairing operation

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b) \,\}$$

where

$$\begin{aligned} &\forall a_1, a_2 \in A, b_1, b_2 \in B. \\ &\quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \end{aligned}$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

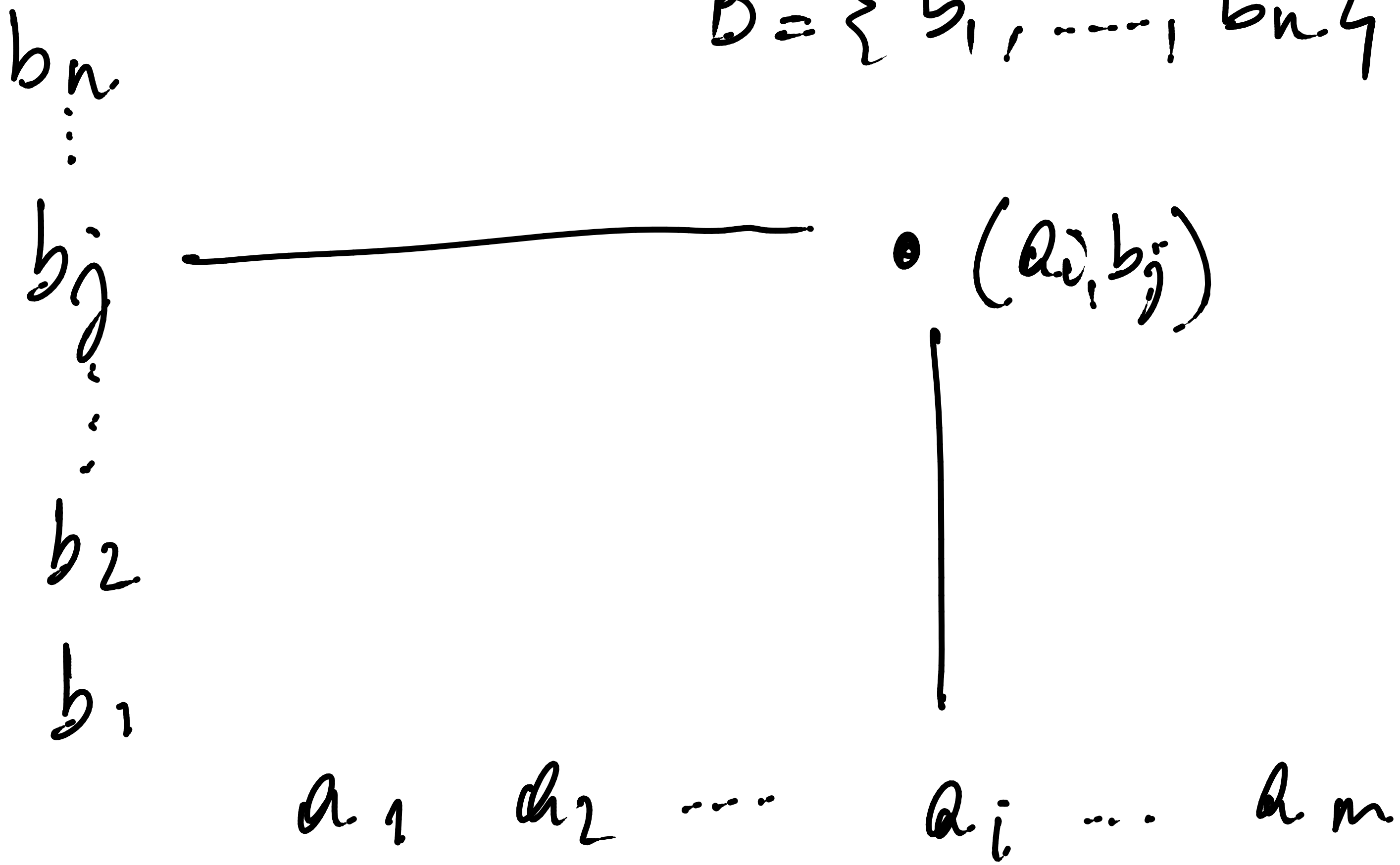
In practice we write:

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

Proposition 89 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA: Suppose $A = \{a_1, \ldots, a_m\}$
$B = \{b_1, \ldots, b_n\}$



$$A \times B = \{ (a_i, b_j) \mid i = 1, \dots, m;\ j = 1 \dots n \}$$

$$\#(A \times B) = m \cdot n = \#A \cdot \#B.$$

$\boxtimes$

Big unions

Definition 90 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$, *we let the* big union *(relative to* $\mathcal{U}$*) be defined as*

Idea

$$\boxed{\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}.\, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .}$$

$$\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U})) \iff \mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$$

$$\mathcal{F} = \{ \ldots, A, A', \ldots, B, \ldots \}$$

where $A, A', \ldots B, \ldots \subseteq \mathcal{U}$

$$\bigcup \mathcal{F} = (\ldots \cup A \cup A' \cup \ldots \cup B \cup \ldots) \subseteq \mathcal{U}$$

Big intersections

Definition 92 *Let* U *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(U)$, *we let the* big intersection *(relative to* U*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in U \mid \forall A \in \mathcal{F}. \, x \in A \} \quad .$$

Idea: $\mathcal{F} = \{ \ldots, A, A', \ldots, B, \ldots \}$

$$\bigcap \mathcal{F} = (\cdots \cap A \cap A' \cap \cdots \cap B \cap \cdots)$$

Theorem 93 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \,\middle|\, (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF:

collects all subsets of $\mathbb{R}$, say S, satisfying a "closure" property; namely, that:

- 0 is in S

and

- whenever x is in S then so is $x+1$.

(i) $\mathbb{N} \in \mathcal{F}$

$\iff$ 0 is in $\mathbb{N}$ and whenever x is in $\mathbb{N}$ so is $x+1$ ✓

$\bigcap \mathcal{F} \subseteq \mathbb{N}$

<u>NB</u> $\forall A \in \mathcal{F}.\ \bigcap \mathcal{F} \subseteq A$ (exercise)

(ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$

(ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$.

We show

$\mathbb{N} \subseteq S$ for all $S \subseteq \mathbb{R}$ satisfying the closure property.

$$\boxed{\forall n \in \mathbb{N}.\ n \in S \quad \text{for all } S \in \mathcal{F}}$$

exercise: prove it by induction.

NB iff

$X \subseteq \bigcap \mathcal{F}$

$X \subseteq A$ for all A in $\mathcal{F}$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \left(\forall X \in \mathcal{F}.\, x \in X\right) \quad .$$

$\{1\}\times A = \{(1,a) \mid a\in A\}$ $\{2\}\times B = \{(2,b) \mid b\in B\}$

NB: $(\{1\}\times A)\cap(\{2\}\times B) = \emptyset$

Disjoint unions

Definition 94 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\}\times A\big) \cup \big(\{2\}\times B\big) \quad .$$

$$= \{(1,a) \mid a\in A\} \cup \{(2,b) \mid b\in B\}$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1,a)\big) \vee \big(\exists b \in B.\, x = (2,b)\big) \,.$$

Idea In ML

datatype

'a 'b disunion

= One of 'a | Two of 'b

Proposition 96 *For all finite sets* A *and* B*,*

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

$A = \{a_1, \ldots, a_m\} \qquad B = \{b_1, \ldots, b_n\}$

$A \cup B = \{\underbrace{a_1, \ldots, a_m}_{m}, \underbrace{b_1, \ldots, b_n}_{n}\}$

$m \quad + \quad n$

Corollary 97 *For all finite sets* A *and* B*,*

$$\#(A \uplus B) = \#A + \#B \quad .$$

A relation from A to B is a set consisting of pairs with first component in A and second component in B.

Relations

Definition 99 *A* (binary) relation R *from a set* A *to a set* B

$$\mathrm{R} : \mathrm{A} \nrightarrow \mathrm{B} \quad \textit{or} \quad \mathrm{R} \in \mathrm{Rel}(\mathrm{A}, \mathrm{B}) \quad ,$$

is

$$\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B} \quad \textit{or} \quad \mathrm{R} \in \mathcal{P}(\mathrm{A} \times \mathrm{B}) \quad .$$

Notation 100 *One typically writes* $a\,\mathrm{R}\,b$ *for* $(a, b) \in \mathrm{R}$.

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

Examples:

- Empty relation.

  $\emptyset : A \mathrel{\longrightarrow\!\!\!\!\!\!\!+\;\;} B$ $\qquad$ ($a \; \emptyset \; b \iff \mathbf{false}$)

- Full relation.

  $(A \times B) : A \mathrel{\longrightarrow\!\!\!\!\!\!\!+\;\;} B$ $\qquad$ ($a \; (A \times B) \; b \iff \mathbf{true}$)

- Identity (or equality) relation.

  $\mathrm{id}_A = \{ \, (a, a) \mid a \in A \, \} : A \mathrel{\longrightarrow\!\!\!\!\!\!\!+\;\;} A$ $\qquad$ ($a \; \mathrm{id}_A \; a' \iff a = a'$)

- Integer square root.

  $R_2 = \{ \, (m, n) \mid m = n^2 \, \} : \mathbb{N} \mathrel{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathbb{Z}$ $\qquad$ ($m \; R_2 \; n \iff m = n^2$)