Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$ (Γ-environments)

$\Gamma = (x_1 : \tau_1, \ldots, x_n : \tau_n)$

$x \in \underline{dom}(\Gamma) \iff x = x_i$ for some $i = 1 - n$

$\Gamma(x) = \tau_i$ if $x \in \underline{dom}(\Gamma) \wedge x = x_i$

$\rho \in [\![\tau_1]\!] \times [\![\tau_2]\!] \times \cdots \times [\![\tau_n]\!]$

$\quad \parallel$

$(d_1, d_2, \ldots, d_n)$ s.t. $d_i \in [\![\tau_i]\!]$ $\forall i$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$ (Γ-environments)

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

$(d_1, d_2, \ldots, d_n) \longleftrightarrow [x_1 \mapsto d_1, x_2 \mapsto d_2, \ldots, x_n \mapsto d_n]$

TUPLES PARTIAL FUNCTIONS

Denotational semantics of PCF type environments

$$\llbracket \Gamma \rrbracket \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} \llbracket \Gamma(x) \rrbracket \qquad (\Gamma\text{-environments})$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in \llbracket \Gamma(x) \rrbracket$ for all $x \in dom(\Gamma)$

Example:

$\Gamma = (x_i : \tau_i)_{i=1-n}$

1. For the empty type environment $\emptyset$,

$\underline{n=0} \Rightarrow \Gamma = (\,)$

$$\llbracket \emptyset \rrbracket = \{ \perp \}$$

where $\perp$ denotes the unique partial function with $dom(\perp) = \emptyset$.

2. $[\![\langle x \mapsto \tau \rangle]\!] = (\{x\} \to [\![\tau]\!]) \cong [\![\tau]\!]$

3.

$$
\begin{aligned}
&[\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
&\quad \cong (\{x_1\} \to [\![\tau_1]\!]) \times \ldots \times (\{x_n\} \to [\![\tau_n]\!]) \\
&\quad \cong [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!] = \mathbb{N}_\perp$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!] = \mathbb{B}_\perp$$

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad (x \in dom(\Gamma))$$

equiv.

$$[\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i]\!](d_1, d_2, \ldots, d_n) = d_i$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$[\![\Gamma \vdash succ(M)]\!] : [\![\Gamma]\!] \longrightarrow \mathbb{N}_\bot$

} strict continuous function

Recall

strict = $\bot$ preserving

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

N.B.
strict

$$[\![\Gamma \vdash M]\!] = 0 \overset{?}{\Longrightarrow} [\![\Gamma \vdash \mathbf{pred}(M)]\!] = \bot$$

by def

$$\frac{M \Downarrow succ(V)}{pred(M) \Downarrow V}$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

For $\rho \in [\![\Gamma]\!]$: $[\![\Gamma \vdash M_1]\!](\rho) : [\![\tau]\!] \to [\![\tau']\!]$

$[\![\Gamma \vdash M_2]\!](\rho) : [\![\tau]\!]$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$\Gamma \quad \lambda \rho \in [\![\Gamma]\!].\ ([\![\Gamma \vdash M_1]\!]\ \rho)(\ [\![\Gamma \vdash M_2]\!]\ \rho)$

$$[\![\Gamma \vdash M_1\ M_2]\!](\rho) \stackrel{\text{def}}{=} ([\![\Gamma \vdash M_1]\!](\rho))\ ([\![\Gamma \vdash M_2]\!](\rho))$$

$\Gamma \vdash M_1 : \tau \to \tau'$ $\qquad [\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \to [\![\tau']\!])$

$\Gamma \vdash M_2 : \tau$ $\qquad [\![\Gamma \vdash M_2 : [\![\Gamma]\!] \to [\![\tau]\!]$

$$[\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \to [\![\tau']\!])$$

define

$$\frac{\Gamma, x:\tau \vdash M : \tau'}{\Gamma \vdash \underline{\mathrm{fn}}\; x:\tau.\, M : \tau \to \tau'}$$

(cf Currying)

by induction

$$[\![\Gamma, x:\tau \vdash M]\!] : \underbrace{[\![\Gamma, x:\tau]\!]}_{[\![\Gamma]\!] \times [\![\tau]\!]} \to [\![\tau']\!]$$

$$\lambda \rho \in [\![\Gamma]\!].\, \underbrace{\Big(\lambda d \in [\![\tau]\!].\, \underbrace{[\![\Gamma, x:\tau \vdash M]\!](\rho, d)}_{[\![\tau']\!]}\Big)}_{[\![\tau]\!] \to [\![\tau']\!]}$$

Denotational semantics of PCF terms, IV

$$\llbracket \Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M \rrbracket(\rho) \qquad\qquad (x \notin dom(\Gamma))$$

$$\stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket \,.\, \llbracket \Gamma[x \mapsto \tau] \vdash M \rrbracket(\rho[x \mapsto d])$$

NB: $\rho[x \mapsto d] \in \llbracket \Gamma[x \mapsto \tau] \rrbracket$ is the function mapping x to $d \in \llbracket \tau \rrbracket$ and otherwise acting like ρ.

$$\frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \underline{fix}(M) : \tau} \quad \begin{array}{l} \leadsto [\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\tau]\!]) \\ \leadsto \forall \rho \in [\![\Gamma]\!] : \; [\![\Gamma \vdash M]\!]\rho : [\![\tau]\!] \to [\![\tau]\!] \end{array}$$

Denotational semantics of PCF terms, V

$$\leadsto \qquad \underline{fix}([\![\Gamma \vdash M]\!]\rho) \in [\![\tau]\!]$$

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$*, the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

$\underline{\mathrm{PCF}}_\tau \stackrel{def}{=} \{ M \mid \ \vdash M : \tau \}$

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{ \perp \}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

$[\![M]\!] \in [\![\tau]\!] \quad \forall M \in \underline{\mathrm{PCF}}_\tau$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

$$\textit{if} \quad [\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\textit{then} \quad [\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M']]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$$

By induction on the structure of $\mathcal{C}[-]$.

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

if $M \Downarrow_\tau V$ *then* $[\![M]\!] = [\![V]\!] \in [\![\tau]\!]$.

By induction on the derivation:

$$M \Downarrow_\tau V$$

$$\frac{M \Downarrow \underline{succ}(V)}{\underline{pred}(M) \Downarrow V}$$

Values of nat are

$0, \underline{succ}(0), \ldots, \underline{succ}^n(0), \ldots$

$[\![\underline{pred}(M)]\!] \overset{?}{=} [\![V]\!]$

By induction, $[\![M]\!] = [\![\underline{succ}(V)]\!]$

$$\begin{cases} m-1 & \text{if } [\![M]\!] = m > 0 \\ \perp & \text{o/w} \end{cases} = \begin{cases} n+1 \\ \perp \end{cases}$$

$[\![V]\!] = n \in \mathbb{N}$

$[\![V]\!] = \perp$

$= [\![V]\!] + 1$

$\| \quad [\![V]\!]$

$$\frac{M_1 \Downarrow \underline{\mathrm{fn}}\, x.\, M \qquad M[{}^{M_2}/_{x}] \Downarrow V}{M_1(M_2) \Downarrow V}$$

By induction:

$$[\![M_1]\!] = [\![\mathrm{fn}\ x.\, M]\!] \qquad [\![M[{}^{M_2}/_{x}]]\!] = [\![V]\!]$$

<u>RTP</u>

$$[\![M_1]\!]([\![M_2]\!]) \overset{?}{=} [\![V]\!]$$

$$\|$$

$$[\![\underline{\mathrm{fn}}\ x.\, M]\!]([\![M_2]\!]) = [\![x \vdash M]\!]([x \mapsto [\![M_2]\!]])$$

$= ?$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*.*

Then,

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)
\end{aligned}
$$

for all $\rho \in [\![\Gamma]\!]$*.*

SYNTACTIC SUBSTITUTION $\longleftrightarrow$ SEMANTIC COMPOSITION.

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$\begin{aligned}&[\![\Gamma \vdash M'[M/x]]\!](\rho)\\&\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)\end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$.

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$