Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

$= \bigsqcup_n f^n(\bot)$

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

$$f \longmapsto \mathit{fix}(f)$$

is continuous.

- fix monotone

$$f \sqsubseteq g : D \to D \implies \underline{\mathrm{fix}}(f) \sqsubseteq \underline{\mathrm{fix}}(g) \in D$$

$$\dfrac{\dfrac{\dfrac{}{f(\mathrm{fix}\, g) \sqsubseteq g(\mathrm{fix}\, g)} \qquad \dfrac{\checkmark}{g(\mathrm{fix}\, g) \sqsubseteq \mathrm{fix}(g)}}{f(\mathrm{fix}\, g) \sqsubseteq \mathrm{fix}\, g}}{\mathrm{fix}(f) \sqsubseteq \mathrm{fix}(g)}$$

- fix preserves lubs of ω-chains.

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \qquad \text{(new)}$$

$$\underline{fix}\Big(\underbrace{\bigsqcup_n f_n}_{\wr \ \text{in } D \to D}\Big) \overset{?}{=} \underbrace{\bigsqcup_n fix(f_n)}_{\wr \ \text{in } D}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\bigsqcup_k \underbrace{\Big(\bigsqcup_n f_n\Big)^k(\bot)}_{\|} \qquad\qquad \bigsqcup_n \bigsqcup_k f_n^{\,k}(\bot)$$

$$\bigsqcup_k \bigsqcup_\ell f_\ell^{\,k}(\bot) \overset{\checkmark}{=\!=\!=\!=} \bigsqcup_\ell f_\ell^{\,\ell}(\bot)$$

$$\left(\bigsqcup_n f_n\right)^k(\bot)$$

$k=0$ $\quad \bot$

$k=1$ $\quad \left(\bigsqcup_n f_n\right)(\bot)$

$\|$

$\bigsqcup_n f_n(\bot)$

$$k=2 \quad \left(\bigsqcup_m f_m\right)\left(\left(\bigsqcup_n f_n\right)(\bot)\right)$$

$$= \left(\bigsqcup_m f_m\right)\left(\bigsqcup_n f_n(\bot)\right)$$

$$= \bigsqcup_m f_m\left(\bigsqcup_n f_n(\bot)\right)$$

$\rightsquigarrow f_m$ continuous

$$= \bigsqcup_m \bigsqcup_n f_m(f_n(\bot))$$

$$= \bigsqcup_\ell f_\ell^2(\bot)$$

$\left(\bigsqcup_k h_k\right)(x)$

$\|$

$\bigsqcup_k \left(h_k(x)\right)$

By induction on k show that

$$\left(\bigsqcup_n f_n\right)^k (\perp) = \bigsqcup_\ell f_\ell^{\,k}(\perp)$$

Topic 4

Scott Induction

We want to show

$f : D \to D$ $\quad P \subseteq D$

$\mathrm{fix}(f) \overset{?}{\in} P$

$\bigsqcup_n f^n(\bot) = \mathrm{fix}(f)$

$\bot \in P \sqsubseteq f(\bot) \in P \sqsubseteq f^2(\bot) \in P \sqsubseteq \cdots$

If $\left(\bigsqcup_n d_n\right) \in P$

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ in P

Then $\mathrm{fix}(f) \in P$

Let P be an <u>admissible</u> property

that is,
$$\bot \in P$$
$$\forall d_0 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \text{ in } P.\ \left(\bigsqcup_n d_n\right) \in P$$

Then

$$\frac{x \in P \Rightarrow f(x) \in P}{\mathit{fix}(f) \in P}\ \begin{matrix} P \\ \underline{\text{admissible}} \end{matrix}$$

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \overset{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

of D is chain-closed.

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ in $\downarrow(d)$

$\equiv \; d_i \sqsubseteq d \;\; \forall i$

$\Rightarrow \bigsqcup_n d_n \sqsubseteq d$

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow\!(d) \overset{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

  of D is chain-closed.

- The subsets

$$\{(x,y) \in D \times D \mid x \sqsubseteq y\}$$

  and

$$\{(x,y) \in D \times D \mid x = y\}$$

  of $D \times D$ are chain-closed.

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies fix(f) \sqsubseteq d$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x \sqsubseteq d]}{f(x) \sqsubseteq f(d)}\; f\text{-monotone} \qquad \dfrac{\checkmark}{f(d) \sqsubseteq d}}{f(x) \sqsubseteq d}}{x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}}{x \in {\downarrow}(d) \Rightarrow f(x) \in {\downarrow}(d)}}{fix(f) \in {\downarrow}(d)}\; {\downarrow}(d)\text{ admissible}}{fix(f) \sqsubseteq d}
$$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in {\downarrow}(d) \quad & \implies \quad x \sqsubseteq d \\ & \implies \quad f(x) \sqsubseteq f(d) \\ & \implies \quad f(x) \sqsubseteq d \\ & \implies \quad f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d)\ .$$

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S \ = \ \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof 1:

$$\bigsqcup_n f^n(\bot) \overset{?}{\sqsubseteq} \bigsqcup_n g^n(\bot)$$

$$\bot \sqsubseteq g\bot \sqsubseteq g^2\bot \sqsubseteq g^3\bot \sqsubseteq \ldots$$

$$\sqcup\!| \qquad \sqcup\!| \qquad \sqcup\!|?\checkmark$$

$$\bot \sqsubseteq f\bot \sqsubseteq f^2\bot \sqsubseteq f^3\bot \sqsubseteq \ldots$$

By induction on $n \in \mathbb{N}$.

$$f^n\bot \sqsubseteq g^n\bot$$

$$ff\bot \sqsubseteq fg\bot$$

$$fg\bot \sqsubseteq gf\bot$$

$$gf\bot \sqsubseteq gg\bot \implies ff\bot \sqsubseteq gg\bot$$

$$
\begin{array}{c}
[fx \sqsubseteq gx] \\ \hline
gfx \sqsubseteq ggx \\
\rotatebox[origin=c]{90}{$\sqsubseteq$} \\
fgx \sqsubseteq ggx \\ \hline
fx \sqsubseteq gx \Rightarrow fgx \sqsubseteq ggx \\ \hline
x \in \{x \mid fx \sqsubseteq gx\} \Rightarrow g(x) \in \{x \mid fx \sqsubseteq gx\}
\end{array}
\qquad \{x \mid f(x) \sqsubseteq g(x)\} \text{ admissible}
$$

$$
\begin{array}{c}
\hline
fix(g) \in \{x \mid f(x) \sqsubseteq g(x)\} \\ \hline\hline
f(fix\, g) \sqsubseteq g(fix\, g)
\end{array}
\qquad
\begin{array}{c}
\checkmark \\ \hline
g(fix\, g) \sqsubseteq fix(g)
\end{array}
$$

$$
\begin{array}{c}
\hline
f(fix\, g) \sqsubseteq fix\, g \\ \hline
fix(f) \sqsubseteq fix(g)
\end{array}
$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \enspace .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \enspace .$$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then
  $$S \cup T \qquad \text{and} \qquad S \cap T$$
  are chain-closed subsets of D.
- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.
- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

Example (III): Partial correctness

Let $\mathcal{F} : \mathit{State} \rightharpoonup \mathit{State}$ be the denotation of

$$\textbf{while}\ X > 0\ \textbf{do}\ (Y := X * Y; X := X - 1)\ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$
$$\Longrightarrow\ \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$ is given by

$$f(w) = \lambda(x, y) \in \mathit{State}. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]{\downarrow} \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \;.$$

Topic 5

PCF

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \\ & \mid & \mathbf{fn}\, x : \tau \,.\, M \mid M\ M \mid \mathbf{fix}(M) \end{array}$$

where $x \in \mathbb{V}$, an infinite set of variables.

Technicality: We identify expressions up to α-conversion of bound variables (created by the $\mathbf{fn}$ expression-former): by definition a PCF term is an α-equivalence class of expressions.

PCF typing relation, $\Gamma \vdash M : \tau$

- Γ is a type environment, *i.e.* a finite partial function mapping variables to types (whose domain of definition is denoted $dom(\Gamma)$)
- M is a term
- τ is a type.

Notation:

$M : \tau$ means M is closed and $\emptyset \vdash M : \tau$ holds.

$\mathrm{PCF}_\tau \overset{\text{def}}{=} \{M \mid M : \tau\}$.

PCF typing relation (sample rules)

$$(:_{\mathrm{fn}}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)$$

$$(:_{\mathrm{app}}) \quad \frac{\Gamma \vdash M_1 : \tau \to \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \tau'}$$

$$(:_{\mathrm{fix}}) \quad \frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \mathbf{fix}(M) : \tau}$$