How do we prove that an element d in a domain D is the lub of an ω-chain

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots \quad (n \in \mathbb{N})$.

① $\forall i \in \mathbb{N}.\ x_i \sqsubseteq d$

② $\forall u \in D.\ (\forall i \in \mathbb{N}.\ x_i \sqsubseteq u) \Rightarrow d \sqsubseteq u$.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

(1) $\bigsqcup \langle d \sqsubseteq d \sqsubseteq \cdots \sqsubseteq d \sqsubseteq \cdots \rangle = d$ ✓

(2) $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_N \sqsubseteq \cdots \sqsubseteq d_i \sqsubseteq \cdots \quad (i \in \mathbb{N})$

fixed natural number

$$\bigsqcup \langle d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_i \sqsubseteq \cdots \rangle$$

$$= \bigsqcup \langle d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots \sqsubseteq \cdots \rangle$$

$$\bigsqcup \langle d_0 \sqsubseteq \cdots \rangle \overset{?}{\sqsubseteq} \bigsqcup \langle d_N \sqsubseteq \cdots \rangle$$

$$\bigsqcup \langle d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots \rangle \overset{?}{\sqsubseteq} \bigsqcup \langle d_0 \sqsubseteq d_1 \sqsubseteq \cdots \rangle$$

We show that

$$d_i \sqsubseteq \bigsqcup \langle d_N \sqsubseteq \cdots \rangle$$

for all $i \in \mathbb{N}$.

Case (1) $i \geq N$: Then

d_i is in $\langle d_N \sqsubseteq \cdots \sqsubseteq \cdots \rangle$

So $d_i \sqsubseteq \bigsqcup \langle d_N \sqsubseteq \cdots \sqsubseteq \cdots \rangle$

How do we show that

$$\bigsqcup_{k \in \mathbb{N}} x_k \sqsubseteq d?$$

We proceed to show

$$x_i \sqsubseteq d$$

$\forall i \in \mathbb{N}$.

Case ② $i < N$

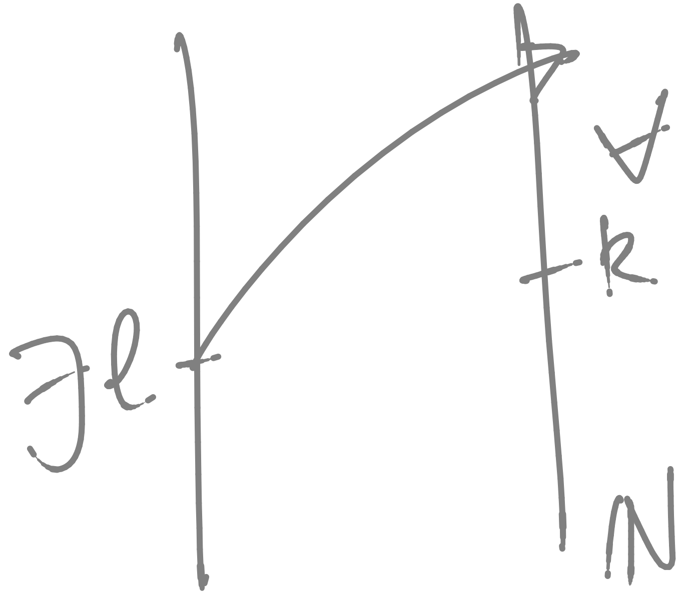
Then $d_i \sqsubseteq d_N \Rightarrow d_i \sqsubseteq \bigsqcup \langle d_N \sqsubseteq \cdots \sqsubseteq \cdots \rangle$.

Fact: For all $\langle d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \rangle$

Consider $f : \mathbb{N} \to \mathbb{N}$ increasing & $(\forall k \in \mathbb{N}.\ \exists l \in \mathbb{N}.\ f(l) > k)$

Then

$$\bigsqcup \langle d_0 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \rangle = \bigsqcup \langle d_{f0} \sqsubseteq d_{f1} \sqsubseteq \cdots \sqsubseteq d_{fn} \sqsubseteq \cdots \rangle$$



$e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots \Longrightarrow \bigsqcup_n e_n$

$d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$

$\bigsqcup_n d_n$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

?

Show that all d_i, $d_i \sqsubseteq \bigsqcup_n e_n$ $\forall i$.

$$\dfrac{\dfrac{\overset{\checkmark}{d_i \sqsubseteq e_i} \qquad \overset{\checkmark}{e_i \sqsubseteq \bigsqcup_n e_n \ \forall i}}{d_i \sqsubseteq \bigsqcup_n e_n \quad \forall i}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$\bigsqcup_m d_0^{(m)} \sqsubseteq \bigsqcup_m d_1^{(m)} \sqsubseteq \cdots \bigsqcup_n \bigsqcup_m d_n^{(m)} \stackrel{?}{=} \bigsqcup_m \bigsqcup_n d_n^{(m)}$$

claim: $\bigsqcup_n \bigsqcup_m d_n^{(m)} = \bigsqcup_k d_k^{(k)} = \bigsqcup_m \bigsqcup_n d_n^{(m)}$

$$\begin{array}{ccccccccccc}
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & \vdots \\
\vdots & & & & & & & & & & \rotatebox{90}{$\sqsubseteq$} \\
\rotatebox{90}{$\sqsubseteq$} & & & & & & & & & & \\
d_0^{(m)} & \sqsubseteq & d_1^{(m)} & \sqsubseteq & \cdots & & \sqsubseteq & d_n^{(m)} & \sqsubseteq & \cdots & \bigsqcup_n d_n^{(m)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & \sqsubseteq & \cdots & & & \vdots & & & \vdots \\
d_0^{(1)} & \sqsubseteq & d_1^{(1)} & \sqsubseteq & d_2^{(1)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(1)} & \sqsubseteq \cdots & \bigsqcup_n d_n^{(1)} \\
\rotatebox{90}{$\sqsubseteq$} & \sqsubseteq & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \cdots & & \rotatebox{90}{$\sqsubseteq$} & \cdots & \rotatebox{90}{$\sqsubseteq$} \\
d_0^{(0)} & \sqsubseteq & d_1^{(0)} & \sqsubseteq & d_2^{(0)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(0)} & \sqsubseteq \cdots & \bigsqcup_n d_n^{(0)}
\end{array}$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{2,n} \;\sqsubseteq\; \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) \; .$$

$\forall i, j:$

$$d_j^{(i)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}$$

because

$$\forall m \quad d_j^{(m)} \sqsubseteq \bigsqcup_n d_n^{m}$$

$$d_j^{(i)} \sqsubseteq \bigsqcup_m d_j^{(m)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{m}$$

$$\frac{\forall k \qquad d_k^{(k)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}}{\bigsqcup_k d_k^{(k)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}}$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\checkmark}{d_n^{(m)} \sqsubseteq d_{\max(m,n)}^{(\max(m,n))} \sqsubseteq \bigsqcup_k d_k^{(k)}}}{\forall n.\ d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}}}{\forall m.\ \bigsqcup_n d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}}}{\bigsqcup_m \bigsqcup_n d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}}
$$

$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

We require $f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n f(d_n)$

Continuity and strictness

always

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

$f(\bigsqcup_n d_n) \sqsupseteq \bigsqcup_n f(d_n)$

D

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \sqsubseteq \bigsqcup_n d_n$

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots f(d_n) \sqsubseteq \cdots \sqsubseteq_n \bigsqcup f(d_n)$

E

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

$\bot \sqsubseteq f(\bot) \Rightarrow f(\bot) \sqsubseteq f^2(\bot) \Rightarrow \bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot)$

$\Rightarrow \cdots \quad \bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots \sqsubseteq f^{(n)}(\bot) \sqsubseteq \cdots \sqsubseteq \bigsqcup_{n \in \mathbb{N}} f^n(\bot)$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

a fix point.

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

FIXED POINT PROPERTY

$$f\left(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)\right) = \bigsqcup_{n\in\mathbb{N}} f(f^n \bot) \qquad f \text{ cont.}$$

$$= \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot)$$

$$= \bigsqcup \langle f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots \rangle$$

$$= \bigsqcup \langle \bot \sqsubseteq f(\bot) \sqsubseteq \cdots \rangle$$

$$= \bigsqcup_{n\in\mathbb{N}} f^n(\bot)$$

$fix(f) =_{df} \bigsqcup_{n \in \mathbb{N}} f^n(\bot)$

is a least pre fixed point.

$$\left[\begin{array}{l} f(fix f) \sqsubseteq fix(f) \\ \forall d.\ f(d) \sqsubseteq d \\ \quad \Rightarrow fix(f) \sqsubseteq d \end{array}\right.$$

Assume $d \in D$ s.t.

$$f(d) \sqsubseteq d$$

RTP:

$$\bigsqcup_n f^n(\bot) \sqsubseteq d$$

by induction

$$\frac{\forall n.\ f^n(\bot) \sqsubseteq d}{\bigsqcup_n f^n(\bot) \sqsubseteq d}$$

$n=0$: $\bot \sqsubseteq d$ ✓

$n>0$: Suppose $f^n(\bot) \sqsubseteq d$

Show $f^{n+1}(\bot) \sqsubseteq d$ $f^{n+1}(\bot) \sqsubseteq f(d)$

$f(d) \sqsubseteq d$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$
\begin{aligned}
&[\![\textbf{while } B \textbf{ do } C]\!] \\
&= \mathit{fix}(f_{[\![B]\!],[\![C]\!]}) \\
&= \textstyle\bigsqcup_{n\geq 0} {f_{[\![B]\!],[\![C]\!]}}^n(\bot) \\
&= \lambda s \in \mathit{State}. \\
&\quad \begin{cases} [\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}
\end{aligned}
$$

NB: $f_{[\![B]\!],[\![C]\!]}$ is continuous!