Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

$\sqsubseteq \subseteq D \times D = \{(d, d') \mid d, d \in D\}.$

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
& & \forall x \in dom(f).\ f(x) = g(x) \\
& \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\ d \sqsubseteq x\ .$$

NB: Suppose d and d' are least elements of D

Then $d = d'$

d is least $d \sqsubseteq x\ \forall x$ so $d \sqsubseteq d'$

d' is least $d' \sqsubseteq x\ \forall x$ so $d' \sqsubseteq d$

$\Rightarrow d = d'$

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.

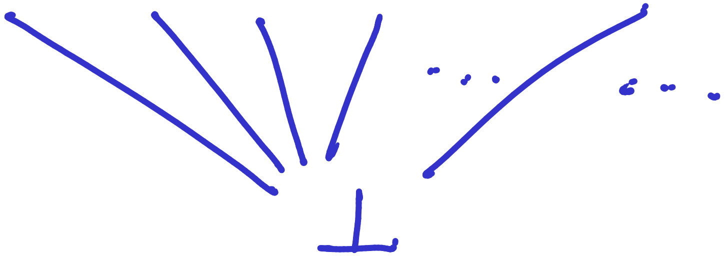
- Note also that a poset may not have least element.

Examples of posets

$$\mathbb{N} = (\{0\ 1\ 2\ 3 \cdots n \cdots\}, =)$$

has no least elements

$$\mathbb{N}_\perp = (0\ 1\ 2\ 3 \cdots n \cdots, \sqsubseteq)$$



$x \sqsubseteq y$ iff $x = \perp$ or else $x = y$

Lifting..

NB $f = id_D : D \to D$ $\quad id_D(x) = x$ $\quad$ Every $d \in D$ is a pre fixed point of id_D.

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{fix(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ fix(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

Suppose d_1 is a least prefixed point of f.

d_2 —— " ——

Then

$d_1 = d_2$

Analogously

⋮

$d_2 \sqsubseteq d_1$

So $d_1 = d_2$.

$\forall d.\ f(d) \sqsubseteq d$

$\Rightarrow d_1 \sqsubseteq d$

In particular d_2 is a prefixed point so

$f(d_2) \sqsubseteq d_2$

Hence

$d_1 \sqsubseteq d_2$

Consider a partial order D with least element $\perp$

$id_D : D \to D$ $\qquad$ $\underline{fix}(id_D)$ exists?

$$\underline{fix}(id_D) = \perp$$

Let $d \in D$ $\quad f = \lambda x . d : D \to D$ $\quad f(x) = d \;\; \forall x$

$\underline{fix}(f)$?

$$\underline{fix}(f) = d$$

Proof principle

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

Proof principle

1.

$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

Claim

$fix(f) = f(fix\,f)$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

f

Show

$$\dfrac{\dfrac{\checkmark\ \mathrm{lfp1}}{f(\mathit{fix}\,f) \sqsubseteq \mathit{fix}(f)} \qquad \dfrac{?}{\mathit{fix}(f) \sqsubseteq f(\mathit{fix}\,f)}}{f(\mathit{fix}\,f) \overset{?}{=} \mathit{fix}\,f}$$

$$\dfrac{x \sqsubseteq y}{f(x) \sqsubseteq f\,y}$$

$$\dfrac{\dfrac{\dfrac{\checkmark\ \mathrm{lfp1}}{f(\mathit{fix}\,f) \sqsubseteq \mathit{fix}(f)}}{f(f(\mathit{fix}\,f)) \sqsubseteq f(\mathit{fix}\,f)}\ f\ \text{mon}}{\mathit{fix}(f) \sqsubseteq f(\mathit{fix}\,f)}$$

Not all monotone functions on a partial order with least element have a least prefixed point.

$$0 \le 1 \le 2 \le \cdots \quad \le n \le \cdots \qquad n \in \mathbb{N}$$

succ

NB Is not a domain!

Thesis*

All domains of computation are
complete partial orders with a least element.

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Thesis*

All domains of computation are complete partial orders with a least element.

All computable functions are continuous.

monotone

+

D

least upper bound

$\exists \bigsqcup_n d_n \in D$

s.t.

- $\forall i \in \mathbb{N}.$ $d_i \sqsubseteq \bigsqcup_n d_n$

- $\forall d \in D.$ $(\forall i \in \mathbb{N}.\ d_i \sqsubseteq d) \Rightarrow \bigsqcup_n d_n \sqsubseteq d$

$n \in \mathbb{N}$

countable chain

$\bot \quad d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

A domain is a cpo that possesses a least element, $\bot$:

$$\forall d \in D \,.\, \bot \sqsubseteq d.$$

$$\frac{}{\perp \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

Proof Principle

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

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$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$
f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}
$$

Least element $\perp$ is the totally undefined partial function.

For a set X, let $\mathcal{P}(X)$ be the powerset of X

$$\mathcal{P}(X) = \{ S \mid S \text{ is a subset of } X \}$$

$(\mathcal{P}(X), \subseteq)$ is a domain

- $\subseteq$ partial order.
- $\emptyset$ is least
- lubs of ω-chains.

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

$n \in \mathbb{N}$

the lub of S_i

$$\geq$$

$$\bigcup_n S_n$$

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

   for all $N \in \mathbb{N}$.