$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\text{while } B \text{ do } C]\!] : \text{State} \rightharpoonup \text{State}$

$\overset{\text{def}}{=}$

$\cdots [\![B]\!] \cdots [\![C]\!]$

sanity check such that

$[\![\text{while } B \text{ do } C]\!](s)$

$= \text{if}\big([\![B]\!]s,$

$[\![\text{while } B \text{ do } C]\!]([\![C]\!]s),$

$s\big)$

A fixed point of a partial function $f : S \rightharpoonup S$ is an element $x \in S$ s.t. $f(x) = x$

$\rightsquigarrow \boxed{?}$ What function is $[\![\text{while } B \text{ do } C]\!]$ a fixed point of?

$$[\![\text{while } B \text{ do } C]\!] = \lambda s.\ \mathrm{if}\big([\![B]\!](s),\ [\![\text{while } B \text{ do } C]\!]([\![C]\!]s),\ s\big)$$

$$\lambda W : \mathrm{State} \to \mathrm{State}.\ \lambda s : \mathrm{State}.\ \mathrm{if}\big([\![B]\!](s),\ W([\![C]\!]s),\ s\big)$$

Fixed point property of $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\mathbf{while}\ B\ \mathbf{do}\ C]\!])$$

where, for each $b : State \to \{true, false\}$ and $c : State \rightharpoonup State$, we define

$$f_{b,c} : (State \rightharpoonup State) \to (State \rightharpoonup State)$$

as

$$f_{b,c} = \lambda w \in (State \rightharpoonup State).\, \lambda s \in State.\, if\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$?

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

domain $(State \to State)$

contains the empty/completely undefined partial function: $\perp$ (i.e. $\perp(s)\uparrow \ \forall s \in \underline{State}$).

$W_0 = \perp$

$W_1 = (\lambda w. \lambda s. \text{if}(\mathcal{B}[\![B]\!]s, w(\mathcal{C}[\![C]\!]s), s))(\perp)$

$= \lambda s. \text{if}(\mathcal{B}[\![B]\!]s, \perp(\mathcal{C}[\![C]\!]s), s)$

$= \lambda s. \begin{cases} \uparrow & \mathcal{B}[\![B]\!](s) = true \\ s & \text{otw} \end{cases}$

$$W_2 = (\lambda W.\lambda s.\, \mathit{if}\,([\![B]\!]s, W([\![C]\!]s), s))\,(W_1)$$

$$= \lambda s.\, \mathit{if}\,([\![B]\!]s, W_1([\![C]\!]s), s)$$

$$= \lambda s. \begin{cases} s & [\![B]\!]s = \mathit{false} \\ W_1([\![C]\!]s) & \mathit{otw} \end{cases}$$

$$= \lambda s. \begin{cases} s & [\![B]\!]s = \mathit{false} \\ [\![C]\!](s) & [\![B]\!]([\![C]\!]s) = \mathit{false} \\ \uparrow & \mathit{otw} \end{cases}$$

more information

$$
\begin{array}{ccccc}
W_0 & \sqsubseteq & W_1 & \sqsubseteq & f_{[\![B]\!],[\![C]\!]}(W_1) = W_2 \\
\parallel & & \parallel & & \parallel \\
\bot & & f_{[\![B]\!],[\![C]\!]}(W_0) & & f^2_{[\![B]\!],[\![C]\!]}(\bot)
\end{array}
$$

$$
\begin{array}{cccccc}
\ldots\ W_n & \sqsubseteq & W_{n+1} & \sqsubseteq \ldots & \bigsqcup_{n \in \mathbb{N}} f^n_{[\![B]\!],[\![C]\!]}(\bot) \\
\parallel & & \parallel & & \\
f^n_{[\![B]\!],[\![C]\!]}(\bot) & & f^{n+1}_{[\![B]\!],[\![C]\!]}(\bot) & &
\end{array}
$$

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$
\begin{aligned}
& {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot) \\
& \quad = \quad \lambda s \in \mathit{State}. \\
& \qquad \begin{cases}
[\![C]\!]^{k}(s) & \text{if } \exists\, 0 \leq k < n.\, [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\
& \text{and } \forall\, 0 \leq i < k.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \\
& \\
\uparrow & \text{if } \forall\, 0 \leq i < n.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true}
\end{cases}
\end{aligned}
$$

$$D \stackrel{\text{def}}{=} (State \rightharpoonup State)$$

- **Partial order $\sqsubseteq$ on D:**

$w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

iff the graph of w is included in the graph of w'.

- **Least element $\bot \in D$ w.r.t. $\sqsubseteq$:**

$\bot$ = totally undefined partial function

= partial function with empty graph

(satisfies $\bot \sqsubseteq w$, for all $w \in D$).

Topic 2

Least Fixed Points

$\bot \sqsubseteq x \;\; \forall x \in D$

information order.

provides no information.

Thesis

All domains of computation are partial orders with a least element.

$\bot \sqsubseteq f(\bot) \Rightarrow f(\bot) \sqsubseteq f^2(\bot) \Rightarrow \cdots \Rightarrow f^n(\bot) \sqsubseteq f^{n+1}(\bot)$

All computable functions are mononotic.

f

$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

fixed point of f?

$\boxed{?} \; \bigsqcup_n f^n(\bot)$?

$\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots \sqsubseteq f^n(\bot) \sqsubseteq \cdots$