Solutions 0

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Question 1. A card is drawn at random from a pack. Event A is 'the card is an ace', event B is 'the card is a spade', event C is 'the card is either an ace, or a king, or a queen, or a jack, or a 10'. Compute the probability that the card has (i) one of these properties, (ii) all of these properties.

(i) The question is unclear. Does it mean "at least one" or "precisely one"? I'll interpret it as "at least"

$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(\text{ace or spade or ace, king, queen, jack, 10}) = \frac{13 + 3 \times 5}{52} = \frac{28}{52} = 7/13$

(ii) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\text{ace \& spade \& (ace, k, q, j, or 10)}) = \mathbb{P}(\text{ace of spades}) = \frac{1}{52}$

Question 2. A biased die has probabilities p, $2p$, $3p$, $4p$, $5p$, $6p$ of throwing 1, 2, 3, 4, 5, 6 respectively. Find p. What is the probability of throwing an even number?

Probabilities sum to 1, so $p(1+2+3+4+5+6) = 1 \Rightarrow p = \frac{1}{21}$.

$\mathbb{P}(\text{even}) = 2p + 4p + 6p = \frac{12}{21}$

Question 3. Consider drawing 2 balls out of a bag of 5 balls: 1 red, 2 green, 2 blue. What is the probability of the second ball drawn from the bag being blue given that the first ball was blue if (i) the first ball is replaced, (ii) the first ball is not replaced?

(i) If it's replaced, it's immaterial what colour the first ball was

$\mathbb{P}(\text{blue}) = 2/5$

(ii) What's left is 1 red, 2 green, 1 blue. $\mathbb{P}(\text{blue}) = 1/4$.

Question 4. Two cards are drawn from a deck of cards. What is the probability of drawing two queens, given that the first card is not replaced?

$\mathbb{P}(\text{first is queen \& second is queen}) = \frac{4}{52} \times \frac{3}{51}$.

Question 5. A screening test is 99% effective in detecting a certain disease when a person has the disease. The test yields a 'false positive' for 1% of healthy persons tested. Suppose 0.1% of the population has the disease. (i) What is the probability that a person whose test is positive has the disease? (ii) What is the probability that a person whose test is negative actually has the disease after all?

We're told: $\mathbb{P}(\text{test +ve} \mid \text{disease}) = 0.99$

$\mathbb{P}(\text{test +ve} \mid \text{healthy}) = 0.01$

$\mathbb{P}(\text{disease}) = 0.001$

(i) $\mathbb{P}(\text{disease} \mid \text{test +ve}) = \frac{\mathbb{P}(\text{disease \& test +ve})}{\mathbb{P}(\text{test +ve})} = \frac{\mathbb{P}(\text{test +ve} \mid \text{disease})\,\mathbb{P}(\text{disease})}{\mathbb{P}(\text{test +ve} \mid \text{disease})\,\mathbb{P}(\text{disease}) + \mathbb{P}(\text{test +ve} \mid \text{healthy})\,\mathbb{P}(\text{healthy})}$

$= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.01 \times 0.999} = 9.0\,\%$

(ii) $\mathbb{P}(\text{disease} \mid \text{test -ve}) = \dfrac{\mathbb{P}(\text{test -ve} \mid \text{disease})\,\mathbb{P}(\text{disease})}{\mathbb{P}(\text{test -ve} \mid \text{disease})\,\mathbb{P}(\text{disease}) + \mathbb{P}(\text{test -ve} \mid \text{healthy})\,\mathbb{P}(\text{healthy})}$

$= \dfrac{(1-0.99)\times 0.001}{(1-0.99)\times 0.001 + (1-0.01)\times 0.999} \approx 0.001\ \%$

Question 6. What is the probability that in a room of r people at least two have the same birthday?

Model: everyone's birthday is independent, drawn uniformly on $\{1, 2, \cdots, 365\}$ days of the year.

$\mathbb{P}(\geq 2 \text{ have same birthday}) = 1 - \mathbb{P}(\text{all have different birthdays})$

$= 1 - \mathbb{P}(\text{2nd diff. to 1st, 3rd diff. to 1st \& 2nd}, \cdots, r\text{th diff. to 1st, 2nd}, \cdots, (r-1)\text{th})$

$= 1 - \dfrac{364}{365} \times \dfrac{363}{365} \times \cdots \times \dfrac{(365-(r-1))}{365} = 1 - \dfrac{365!}{(365-r)!\ 365^r}$

Question 7. Out of 10 physics professors and 12 chemistry professors, a committee of 5 people must be chosen in which each subject has at least 2 representatives. In how many ways can this be done?

Ignoring the constraints, there are $\binom{22}{5}$ possible committees.

How many of them have
- 0 physics? $\binom{12}{5}$
- 1 physics? $10 \times \binom{12}{4}$
- 0 chemistry? $\binom{10}{5}$
- 1 chemistry? $12 \times \binom{10}{4}$

and these are all disjoint.

So, total number of acceptable committees is $\binom{22}{5} - \left(\binom{12}{5} + 10\binom{12}{4} + \binom{10}{5} + 12\binom{10}{4}\right) = 17820$

Question 8. What is the probability of throwing exactly 3 heads out of 6 tosses of a fair coin? How about at least one head?

(i) $\binom{6}{3} \times \left(\frac{1}{2}\right)^6$

(ii) $\mathbb{P}(\#\text{heads} \geq 1) = 1 - \mathbb{P}(\#\text{heads} = 0) = 1 - \left(\frac{1}{2}\right)^6$.

Question 9. A bag contains 6 blue balls and 4 red balls. Three balls are drawn from the bag without replacement. Let X be the number of these three balls that are red. Find the density function $f(x) = \mathbb{P}(X = x)$.

$\mathbb{P}(X=0) = \dfrac{6\cdot 5\cdot 4}{10\cdot 9\cdot 8}$

$\mathbb{P}(X=1) =$ RBB, BRB or BBR. $\dfrac{4\cdot 6\cdot 5 + 6\cdot 4\cdot 5 + 6\cdot 5\cdot 4}{10\cdot 9\cdot 8} = 3\times\dfrac{6\cdot 5\cdot 4}{10\cdot 9\cdot 8}$

$\mathbb{P}(X=2) =$ RRB, RBR, or BRR: $\dfrac{4\cdot 3\cdot 6 + 4\cdot 6\cdot 3 + 6\cdot 4\cdot 3}{10\cdot 9\cdot 8} = 3\times\dfrac{6\cdot 4\cdot 3}{10\cdot 9\cdot 8}$

$\mathbb{P}(X=3) = \dfrac{4\cdot 3\cdot 2}{10\cdot 9\cdot 8}$

Question 10. Find the mean and variance of the Exponential distribution, which has density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that X takes a value in excess of two standard deviations from the mean?

Mean: $\mathbb{E}X = \int_0^\infty x \lambda e^{-\lambda x}\,dx = \left[-xe^{-\lambda x}\right]_0^\infty - \int_0^\infty (-e^{-\lambda x})\,dx$ (integration by parts)

$= \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda}.$

Variance. $\operatorname{Var} X = \mathbb{E}\left(X-\frac{1}{\lambda}\right)^2 = \int_0^\infty \left(x-\frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x}\,dx$

$= \left[\left(x-\frac{1}{\lambda}\right)^2(-e^{-\lambda x})\right]_0^\infty - \int_0^\infty (-e^{-\lambda x})\, 2\left(x-\frac{1}{\lambda}\right)dx$

$= \frac{1}{\lambda^2} + 2\int_0^\infty xe^{-\lambda x}\,dx - \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}\,dx$

$= \frac{1}{\lambda^2} + \frac{2}{\lambda^2} + \frac{2}{\lambda^2}\left[e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda^2} + \frac{2}{\lambda^2} - \frac{2}{\lambda^2} = \frac{1}{\lambda^2}.$

Prob: $\mathbb{P}\left(X > \frac{1}{\lambda} + \frac{2}{\lambda}\right) = \mathbb{P}\left(X > \frac{3}{\lambda}\right) = \int_{3/\lambda}^\infty \lambda e^{-\lambda x} = \left[-e^{-\lambda x}\right]_{3/\lambda}^\infty = e^{-3}$

Question 11. Players A and B roll a six-sided die in turn. If a player rolls 1 or 2 that player wins and the game ends; if a player rolls 3 the other player wins and the game ends; otherwise the turn passes to the other player. A has the first roll. What is the probability (i) that B gets a first throw and wins on it? (ii) that A wins before A's second throw? (iii) that A wins, if the game is played until there is a winner?

(i) $\mathbb{P}$(B gets a first throw and wins on it)

$= \mathbb{P}$(A rolls 4/5/6 then B rolls 1/2) $= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

(ii) $\mathbb{P}$(A wins before A's second throw) ?

A rolls 1/2

or A rolls 4/5/6 then B rolls 3 $\quad \mathbb{P} = \frac{1}{3} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{12}$

(iii) Let $\pi_A = \mathbb{P}$(A wins eventually | A's turn)

$\pi_B = \mathbb{P}$(A wins eventually | B's turn).

$\pi_A = \frac{1}{3} + \frac{1}{2}\pi_B$ (A wins straight away, or play passes to B and then A wins)

$\pi_B = \frac{1}{6} + \frac{1}{2}\pi_A$ (B throws 3 so A wins, or play passes to A then A wins).

Note: I'm using the Law of Total Probability, and Memorylessness.

Solving these simultaneously,

$\pi_A = \frac{1}{3} + \frac{1}{2}\left(\frac{1}{6} + \frac{1}{2}\pi_A\right) = \frac{5}{12} + \frac{\pi_A}{4} \Rightarrow 3\frac{\pi_A}{4} = \frac{5}{12} \Rightarrow \pi_A = \frac{5}{12} \times \frac{4}{3} = \frac{5}{9}$

Question 12. Players A and B take turns to toss a coin, stopping as soon as B has 6 heads. How many ways could A end up with one head? two heads?

The question is not well-put — there are infinitely many ways (counting all the tails).

But we can still enumerate them! I'll use regular expression syntax.

The question doesn't say who goes first. Let's assume A.

For A to get 1 head:

$$(t_A t_B)^* H_A \underbrace{(t_B t_A)^* H_B t_A}_{5 \text{ times}} (t_B t_A)^* H_B$$

or $$(t_A t_B)^* t_A H_B (t_A t_B)^* H_A \underbrace{(t_B t_A)^* H_B t_A}_{4 \text{ times}} (t_B t_A)^* H_B$$

or

These are all elaborations on the sequence of heads:

$$H_A \underbrace{H_B}_{6 \text{ times}} \quad \text{or} \quad H_B H_A \underbrace{H_B}_{5 \text{ times}} \quad \text{or} \cdots \quad \text{or} \quad \underbrace{H_B}_{5 \text{ times}} H_A H_B$$

but with "parity" tails and "repetition" tails filled in as appropriate.

The rule is: if the head sequence starts H_A, insert $(t_A t_B)^*$ in front
H_B $(t_A t_B)^* t_A$ in front

if the head sequence has $H_A H_B$, insert $(t_B t_A)^*$ in between
$H_B H_A$ $(t_A t_B)^*$ in between
$H_B H_B$ $(t_A t_B)^* t_A$ in between
$H_A H_A$ $(t_B t_A)^* t_B$ in between

For A to get 2 heads:

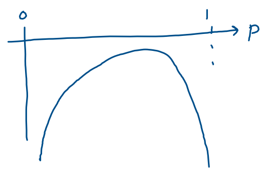
the head sequence must be (two H_A, 5 H_B in any order) then H_B

and there are $\binom{7}{2}$ such choices.

Question 13. Find the value of $p \in [0,1]$ that maximizes $p^a(1-p)^b$, where a and b are both positive. [Hint: take logarithms first.]

let $\quad lik(p) = p^a (1-p)^b$.

$\quad loglik(p) = a \log p + b \log(1-p)$



$$\frac{d}{dp}: \quad \frac{a}{p} - \frac{b}{1-p}$$

The maximum is at $\frac{d}{dp} = 0$: $\quad \frac{a}{\hat{p}} - \frac{b}{1-\hat{p}} = 0 \Rightarrow \frac{a}{\hat{p}} = \frac{b}{1-\hat{p}} \Rightarrow a - a\hat{p} = b\hat{p} \Rightarrow \hat{p} = \frac{a}{a+b}$.

We could also double-check the second derivative:

$$\frac{d^2}{dp^2} = -\frac{a}{p^2} - \frac{b}{(1-p)^2}$$ which is < 0 for all p, hence the function is strictly concave,

so $\frac{d}{dp} = 0$ must be a maximum.