# V. Approximation Algorithms via Exact Algorithms

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Easter 2018

UNIVERSITY OF CAMBRIDGE

# Outline

The Subset-Sum Problem

Parallel Machine Scheduling

# The Subset-Sum Problem

**The Subset-Sum Problem**

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer $t$
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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This problem is NP-hard

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$t = 13$ tons

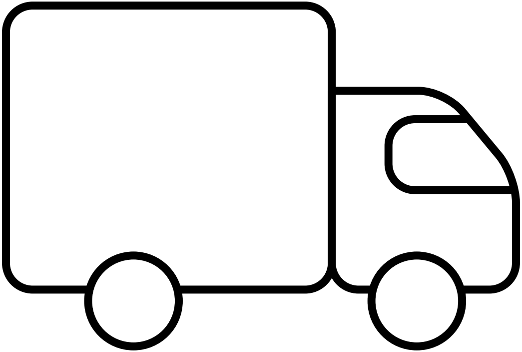
$x_1 = 10$

$x_2 = 4$

$x_3 = 5$

$x_4 = 6$

$x_5 = 1$



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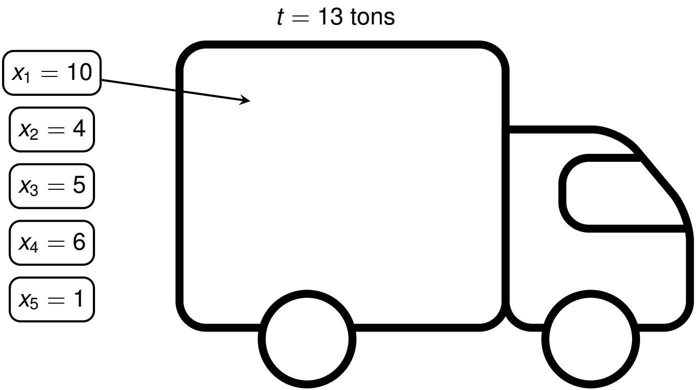

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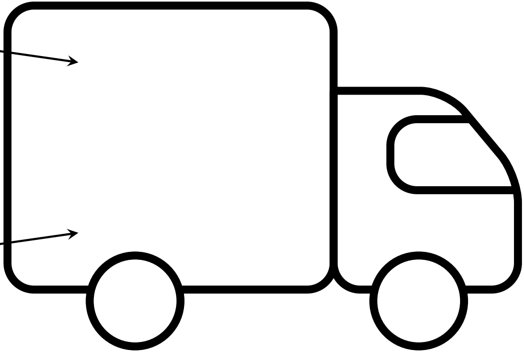

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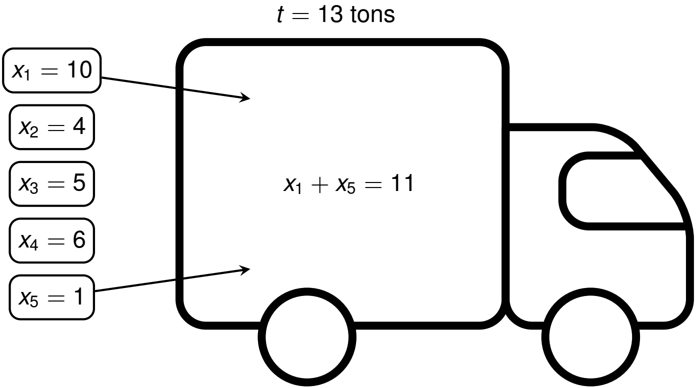

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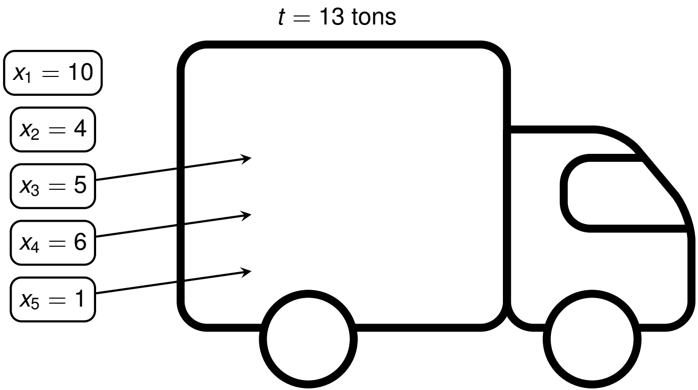

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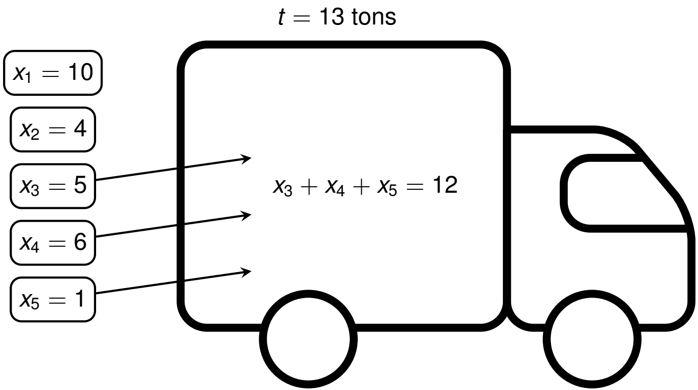

# An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

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$\text{EXACT-SUBSET-SUM}(S, t)$

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n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
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    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$    $S + x := \{s + x : s \in S\}$
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Returns the merged list (in sorted order and without duplicates)

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implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

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can be shown by induction on $n$

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Better runtime if $t$ and/or $|L_i|$ are small.

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```
TRIM(L, δ)
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
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TRIM works in time $\Theta(m)$, if $L$ is given in sorted order.

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
*last* $= y_1$
**for** $i = 2$ **to** $m$
    **if** $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because $L$ is sorted
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last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15 \rangle$$

# Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)      // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20 \rangle$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20 \rangle$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
*last* $= y_1$
**for** $i = 2$ **to** $m$
  **if** $y_i > last \cdot (1 + \delta)$  // $y_i \geq last$ because $L$ is sorted
    append $y_i$ onto the end of $L'$
    *last* $= y_i$
**return** $L'$

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, \overset{\substack{\text{last}\\ \downarrow}}{20}, \underset{\substack{\uparrow \\ \text{i}}}{21}, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20 \rangle$$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
**for** $i = 2$ **to** $m$
  **if** $y_i > last \cdot (1 + \delta)$  // $y_i \geq last$ because $L$ is sorted
    append $y_i$ onto the end of $L'$
    $last = y_i$
**return** $L'$

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20 \rangle$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
**for** $i = 2$ **to** $m$
  **if** $y_i > last \cdot (1 + \delta)$  // $y_i \geq last$ because $L$ is sorted
    append $y_i$ onto the end of $L'$
    $last = y_i$
**return** $L'$

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
**for** $i = 2$ **to** $m$
    **if** $y_i > last \cdot (1 + \delta)$  // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
**return** $L'$

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
**for** $i = 2$ **to** $m$
  **if** $y_i > last \cdot (1 + \delta)$  // $y_i \geq last$ because $L$ is sorted
    append $y_i$ onto the end of $L'$
    $last = y_i$
**return** $L'$

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

# Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

# Illustration of the Trim Operation

TRIM$(L, \delta)$

let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
**for** $i = 2$ **to** $m$
  **if** $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because $L$ is sorted
    append $y_i$ onto the end of $L'$
    $last = y_i$
**return** $L'$

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$

# The FPTAS

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

# The FPTAS

APPROX-SUBSET-SUM$(S, t, \epsilon)$

$n = |S|$
$L_0 = \langle 0 \rangle$
**for** $i = 1$ **to** $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
**return** $z^*$

EXACT-SUBSET-SUM$(S, t)$

$n = |S|$
$L_0 = \langle 0 \rangle$
**for** $i = 1$ **to** $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
**return** the largest element in $L_n$

# The FPTAS

APPROX-SUBSET-SUM$(S, t, \epsilon)$

$n = |S|$
$L_0 = \langle 0 \rangle$
**for** $i = 1$ **to** $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
**return** $z^*$

Repeated application of TRIM to make sure $L_i$'s remain short.

EXACT-SUBSET-SUM$(S, t)$

$n = |S|$
$L_0 = \langle 0 \rangle$
**for** $i = 1$ **to** $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
**return** the largest element in $L_n$

# The FPTAS

APPROX-SUBSET-SUM$(S, t, \epsilon)$

$n = |S|$<br>
$L_0 = \langle 0 \rangle$<br>
**for** $i = 1$ **to** $n$<br>
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$<br>
    $\boxed{L_i = \text{TRIM}(L_i, \epsilon/2n)}$<br>
    remove from $L_i$ every element that is greater than $t$<br>
let $z^*$ be the largest value in $L_n$<br>
**return** $z^*$

Repeated application of TRIM to make sure $L_i$'s remain short.

EXACT-SUBSET-SUM$(S, t)$

$n = |S|$<br>
$L_0 = \langle 0 \rangle$<br>
**for** $i = 1$ **to** $n$<br>
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$<br>
    remove from $L_i$ every element that is greater than $t$<br>
**return** the largest element in $L_n$

- We must bound the inaccuracy introduced by repeated trimming

# The FPTAS

APPROX-SUBSET-SUM$(S, t, \epsilon)$

$n = |S|$
$L_0 = \langle 0 \rangle$
**for** $i = 1$ **to** $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $\boxed{L_i = \text{TRIM}(L_i, \epsilon/2n)}$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
**return** $z^*$

Repeated application of TRIM to make sure $L_i$'s remain short.

EXACT-SUBSET-SUM$(S, t)$

$n = |S|$
$L_0 = \langle 0 \rangle$
**for** $i = 1$ **to** $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
**return** the largest element in $L_n$

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

# The FPTAS

APPROX-SUBSET-SUM$(S, t, \epsilon)$

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

Repeated application of TRIM to make sure $L_i$'s remain short.

EXACT-SUBSET-SUM$(S, t)$

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Solution is a careful choice of $\delta$!

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ϵ)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ϵ/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

# Running through an Example

APPROX-SUBSET-SUM$(S, t, \epsilon)$

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

# Running through an Example

APPROX-SUBSET-SUM$(S, t, \epsilon)$

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4       L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5       L_i = TRIM(L_i, ε/2n)
6       remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$

# Running through an Example

APPROX-SUBSET-SUM$(S, t, \epsilon)$

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$

# Running through an Example

APPROX-SUBSET-SUM$(S, t, \epsilon)$

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$

# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

# Running through an Example

APPROX-SUBSET-SUM$(S, t, \epsilon)$

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
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# Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
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Returned solution $z^* = 302$, which is 2% within the optimum $307 = 104 + 102 + 101$

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**Theorem 35.8**

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

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$$\frac{y^*}{z} \leq \left(1 + \frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1 + \frac{\epsilon/2}{n}\right)^n \stackrel{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

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$$\leq 1+\epsilon/2+(\epsilon/2)^2 \leq 1+\epsilon$$

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$$\begin{aligned}\log_{1+\epsilon/(2n)} t + 2 &= \frac{\ln t}{\ln(1+\epsilon/(2n))} + 2 \\ &\leq \frac{2n(1+\epsilon/(2n)) \ln t}{\epsilon} + 2\end{aligned}$$

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Need $\log(t)$ bits to represent $t$ and $n$ bits to represent $S$

# Concluding Remarks

**The Subset-Sum Problem**

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer $t$
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.

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A more general problem than Subset-Sum

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Algorithm very similar to APPROX-SUBSET-SUM

Theorem

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# Outline

The Subset-Sum Problem

Parallel Machine Scheduling

# Parallel Machine Scheduling

Machine Scheduling Problem

- Given: $n$ jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and $m$ identical machines $M_1, M_2, \ldots, M_m$

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- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \leq j \leq n} C_j$, where $C_k$ is the completion time of job $J_k$.

# Parallel Machine Scheduling

Machine Scheduling Problem

- Given: $n$ jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and $m$ identical machines $M_1, M_2, \ldots, M_m$
- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \leq j \leq n} C_j$, where $C_k$ is the completion time of job $J_k$.

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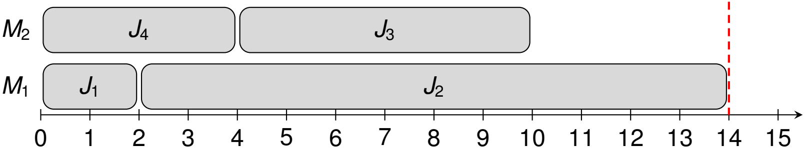

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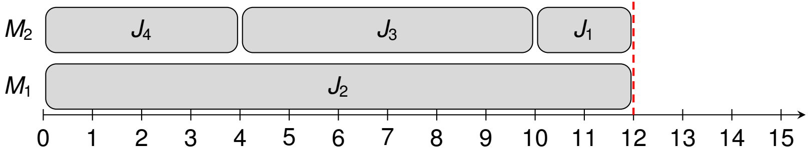

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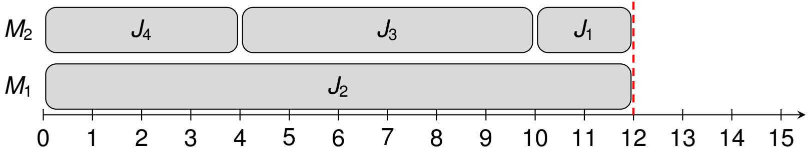
For the analysis, it will be convenient to denote by $C_i$ the completion time of a machine $i$.



## NP-Completeness of Parallel Machine Scheduling

**Lemma**

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

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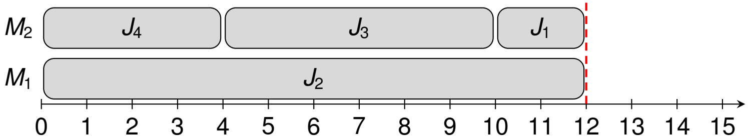

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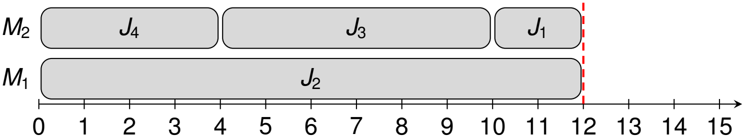


LIST SCHEDULING$(J_1, J_2, \ldots, J_n, m)$

**while** there exists an unassigned job

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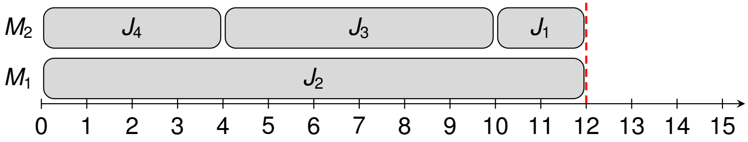


Equivalent to the following Online Algorithm [CLRS]:
Whenever a machine is idle, schedule any job that has not yet been scheduled.

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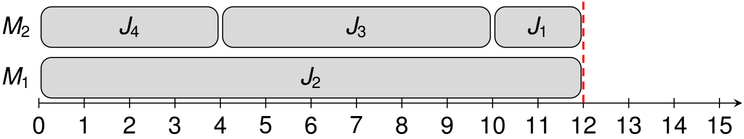


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How good is this most basic Greedy Approach?

# List Scheduling Analysis (Observations)

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Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

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Proof:

b. The total processing times of all $n$ jobs equals $\sum_{k=1}^{n} p_k$

$\Rightarrow$ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ $\square$

# List Scheduling Analysis (Final Step)

**Ex 35-5 d. (Graham 1966)**

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

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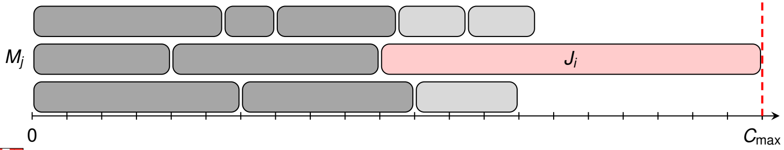

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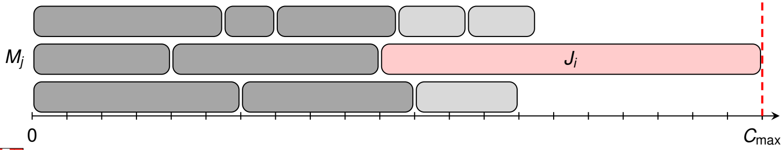

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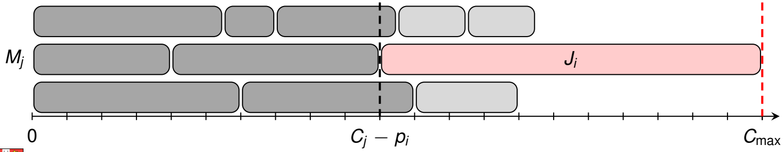

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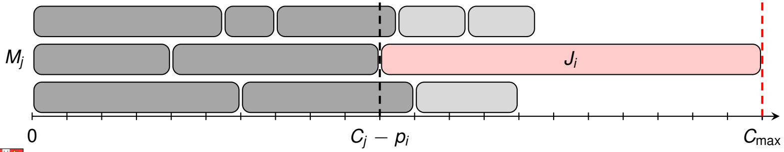

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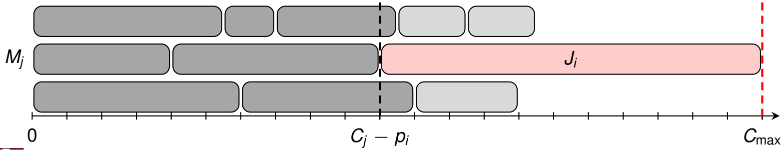

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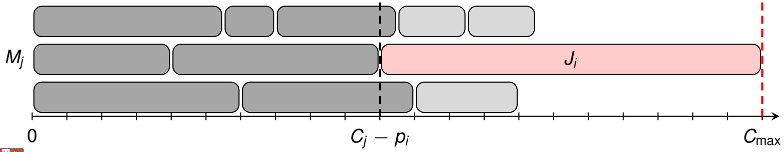

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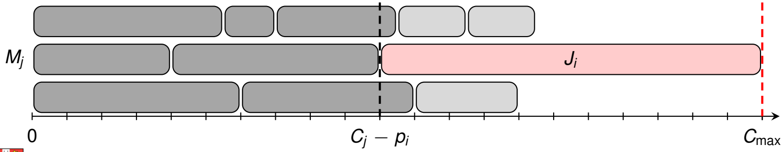

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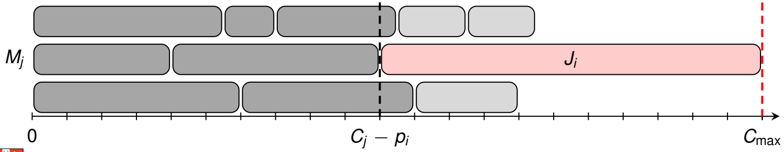

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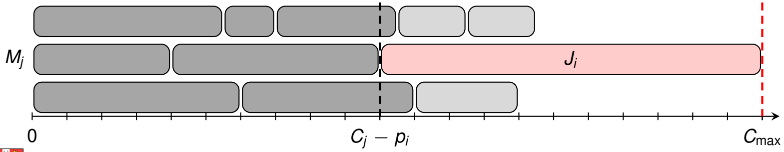

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Analysis can be shown to be almost tight. Is there a better algorithm?

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LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)

1: Sort jobs decreasingly in their processing times
2: **for** $i = 1$ to $m$
3: $\quad C_i = 0$
4: $\quad S_i = \emptyset$
5: **end for**
6: **for** $j = 1$ to $n$
7: $\quad i = \mathrm{argmin}_{1 \le k \le m} C_k$
8: $\quad S_i = S_i \cup \{j\}, C_i = C_i + p_j$
9: **end for**
10: **return** $S_1, \ldots, S_m$

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Runtime:

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- $O(n \log m)$ for extracting (and re-inserting) the minimum (use priority queue).

## Analysis of Improved Greedy

**Graham 1966**

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

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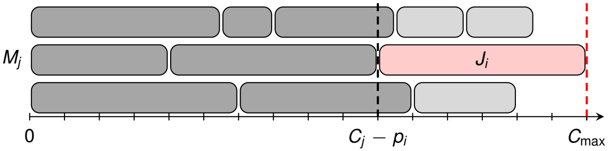

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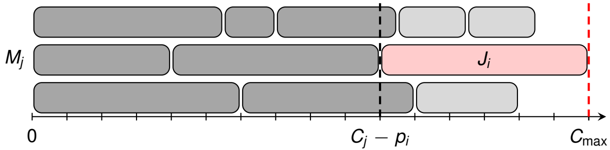

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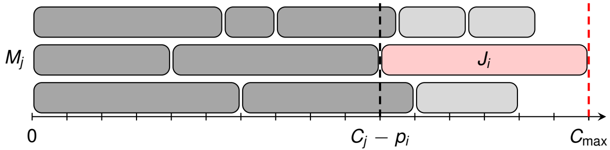
$$C_{\max} = C_j = (C_j - p_i) + p_i \leq C^*_{\max} + \frac{1}{2} C^*_{\max}$$

This is for the case $i \geq m+1$ (otherwise, an even stronger inequality holds)



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- As in the analysis for list scheduling, we have

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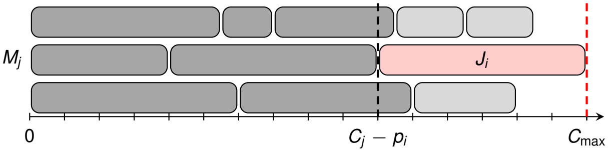

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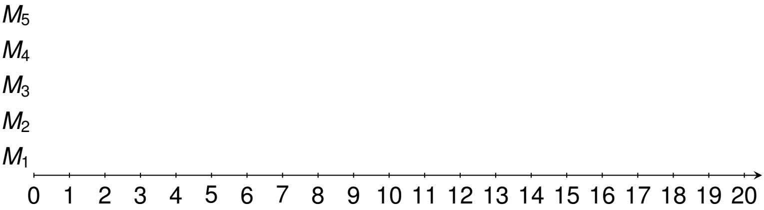

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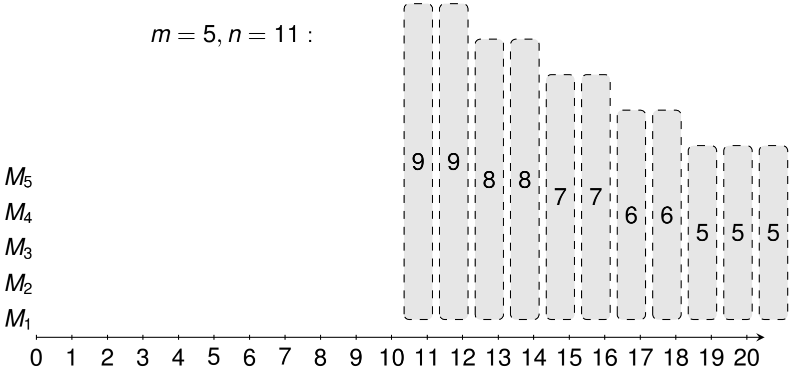

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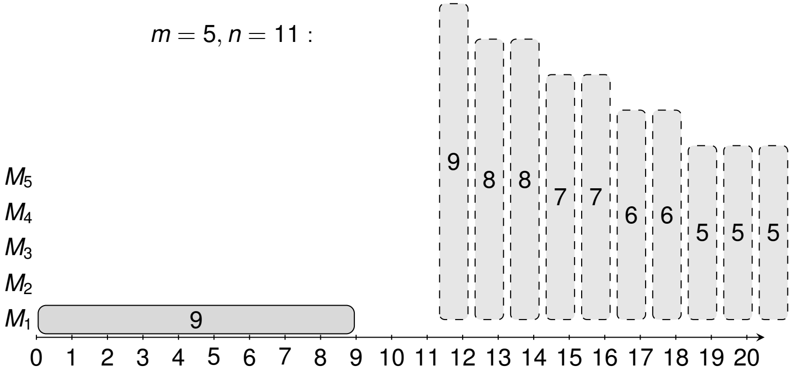

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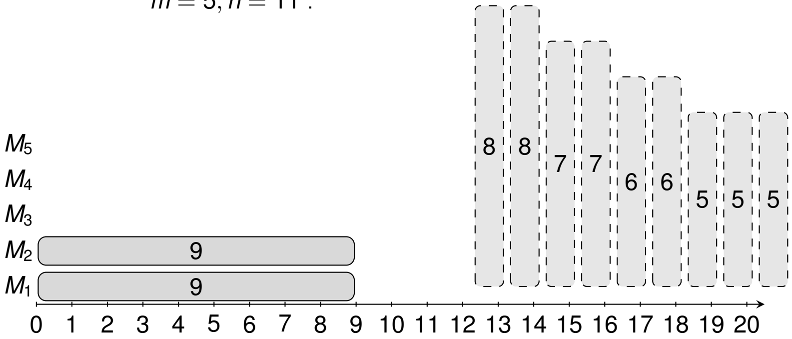

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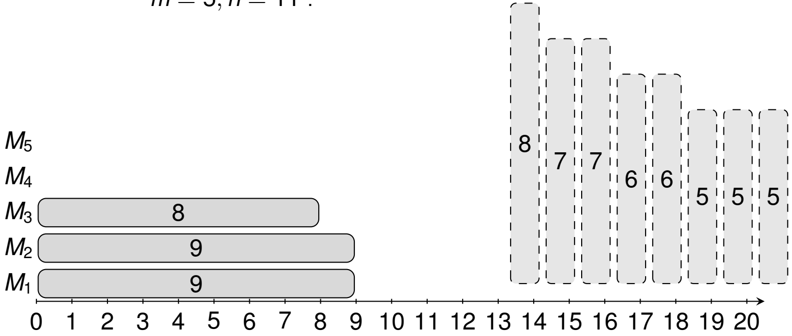

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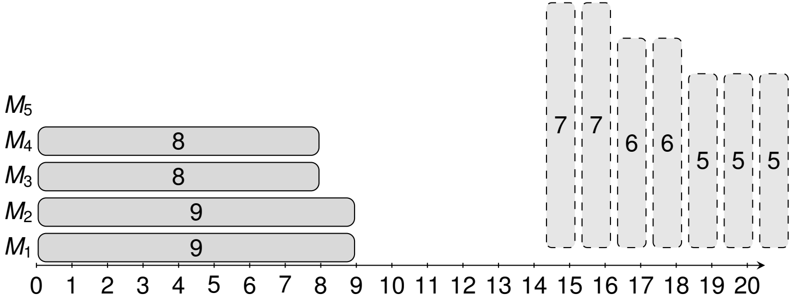

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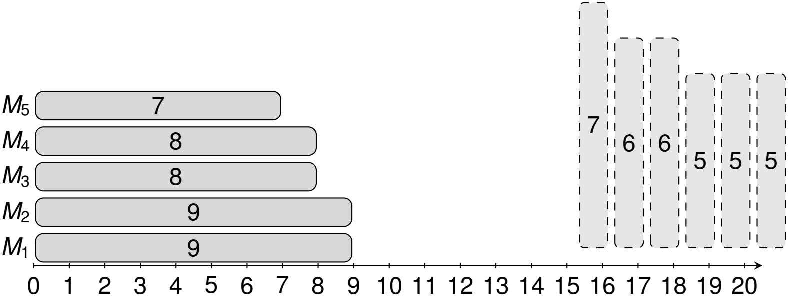

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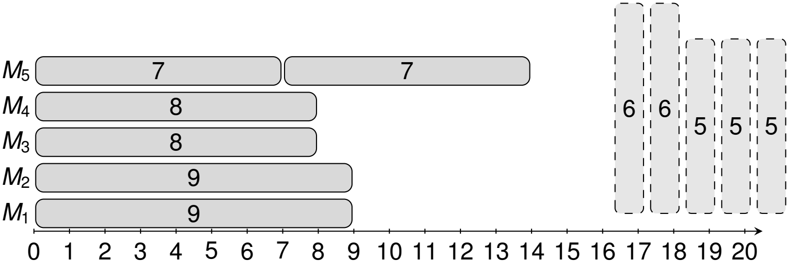

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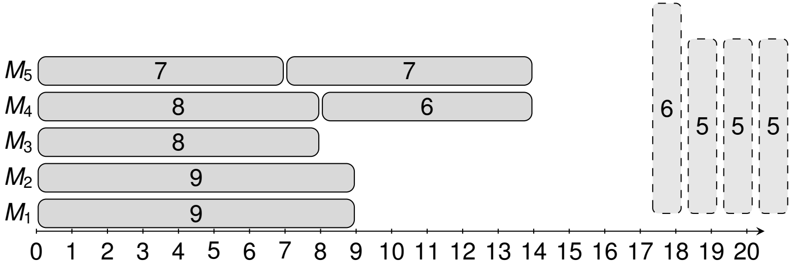

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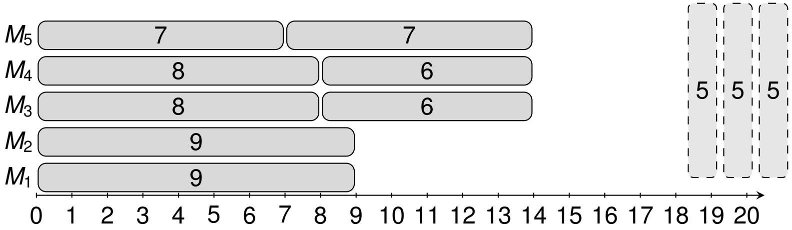

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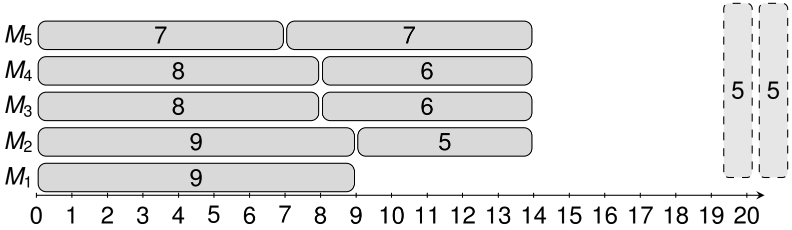

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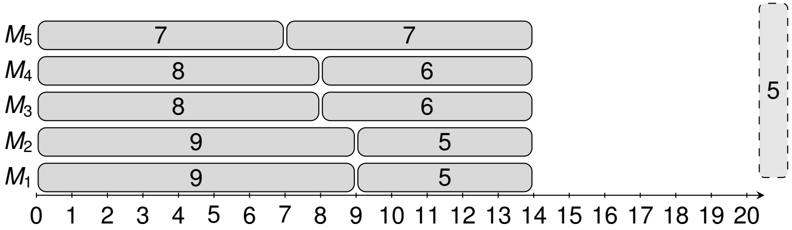

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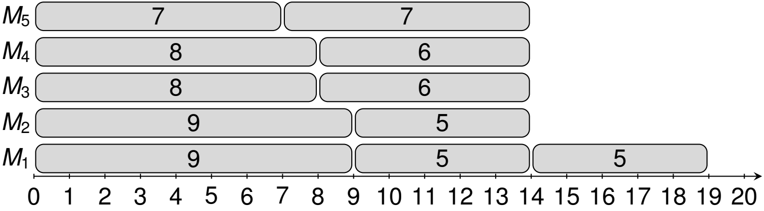

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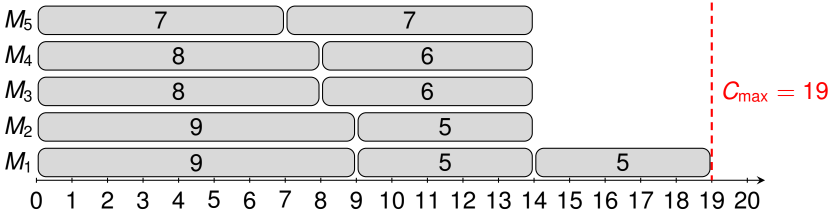

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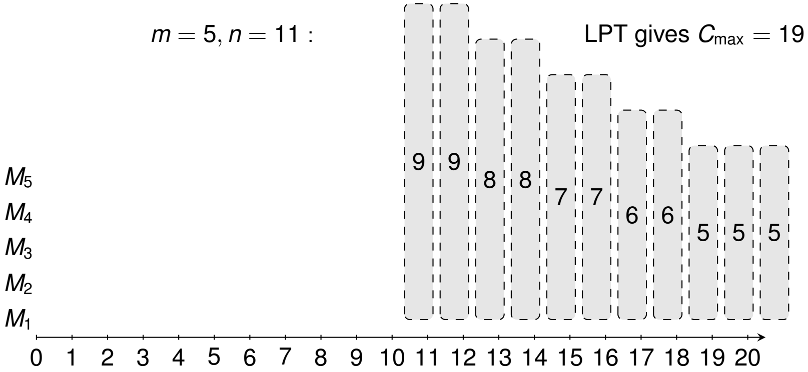

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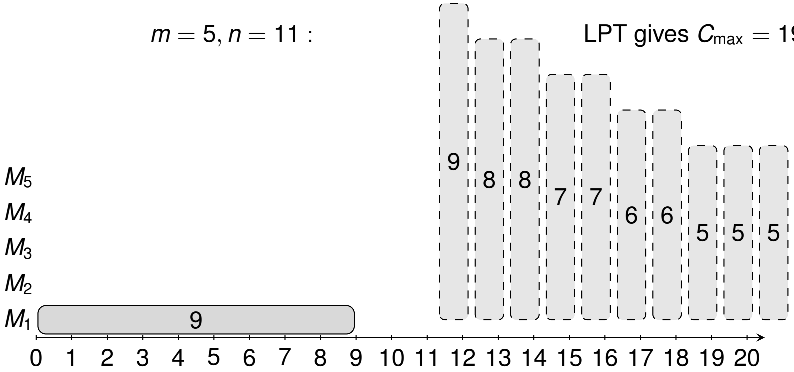

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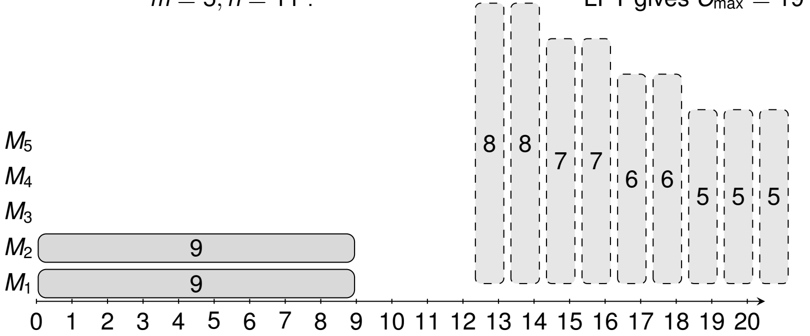

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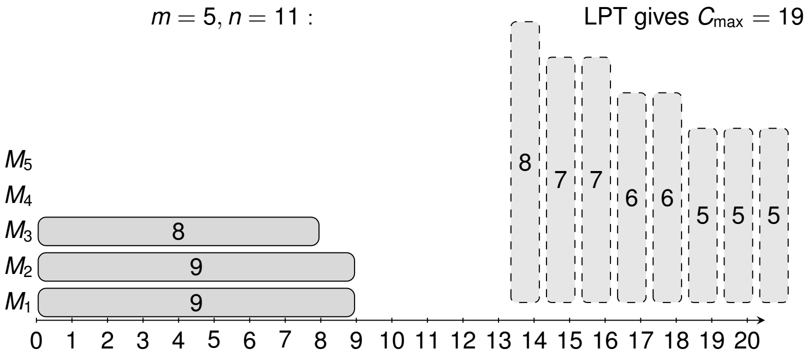

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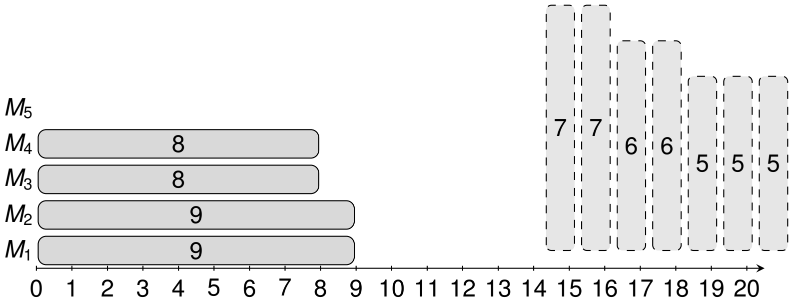

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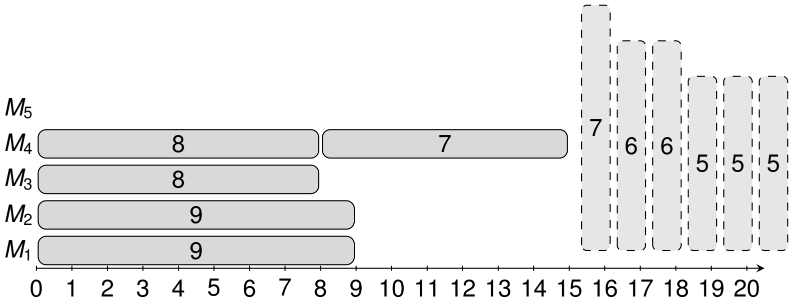

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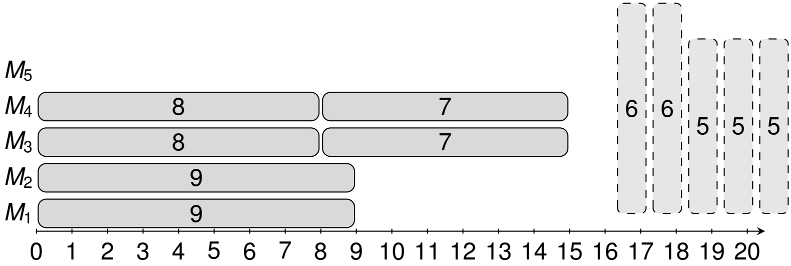

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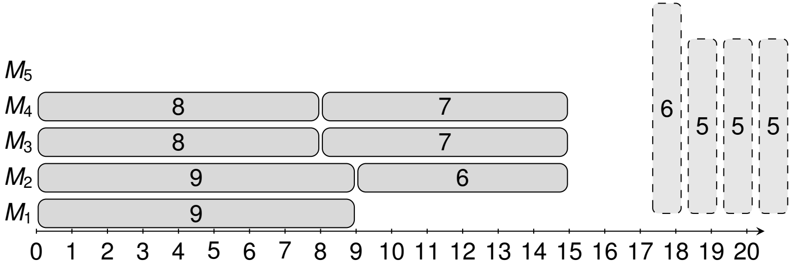

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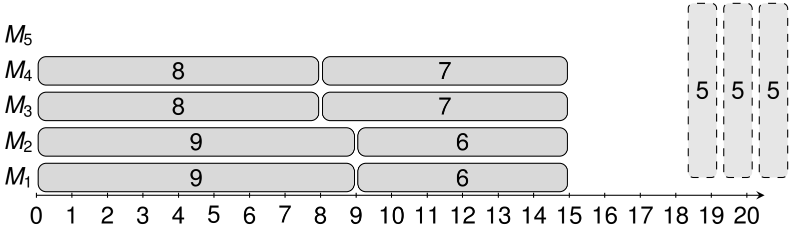

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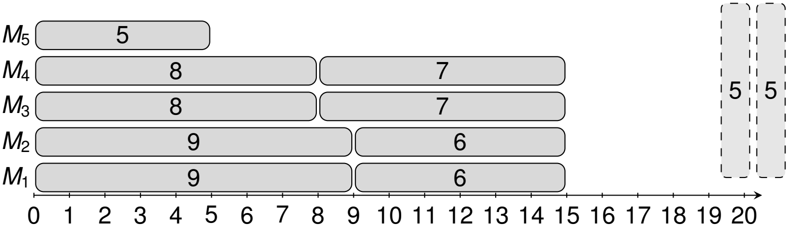

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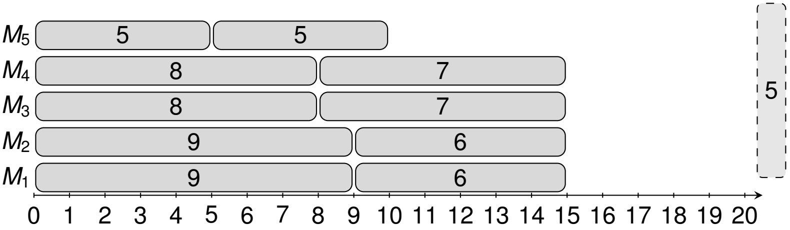

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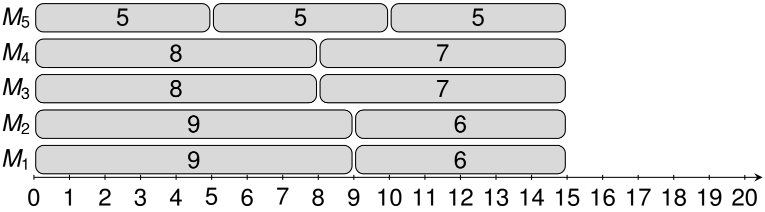

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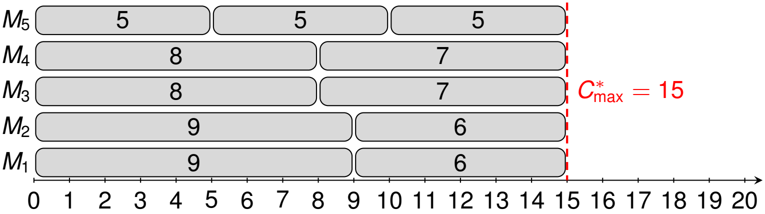

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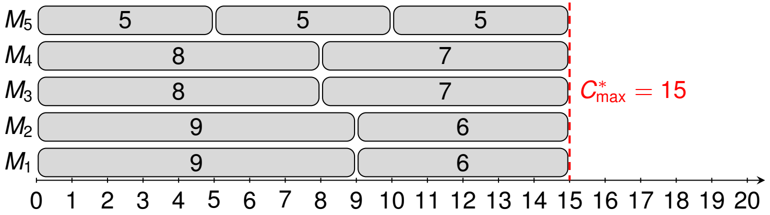

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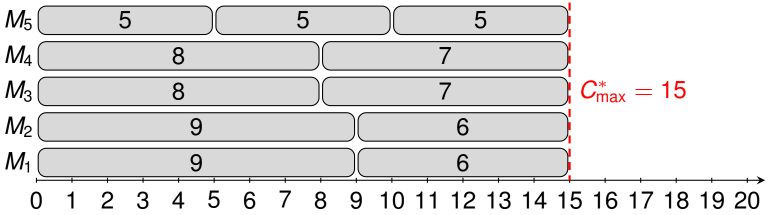

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There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

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Since $0 \leq C^*_{\max} \leq P$ and $C^*_{\max}$ is integral, binary search terminates after $O(\log P)$ steps.

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# A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact $p_k$'s.

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We will prove this on the next slides.

**Key Lemma**

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

**Theorem (Hochbaum, Shmoys'87)**

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Since $0 \leq C^*_{\max} \leq P$ and $C^*_{\max}$ is integral, binary search terminates after $O(\log P)$ steps.

Proof (using Key Lemma):

PTAS$(J_1, J_2, \ldots, J_n, m)$

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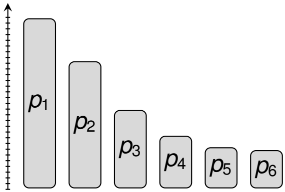

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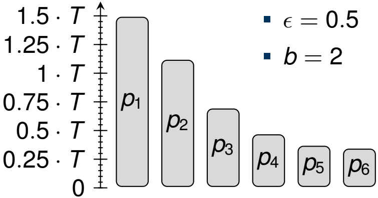


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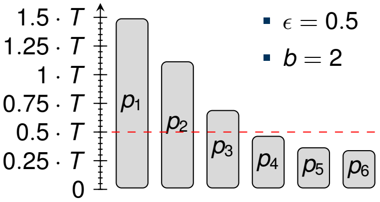


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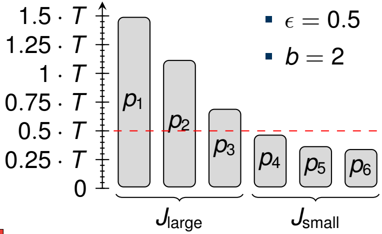


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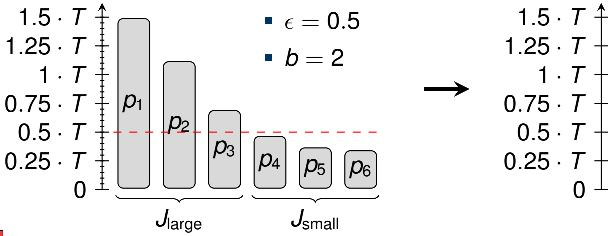

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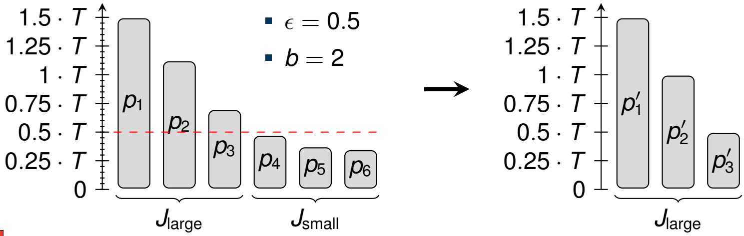

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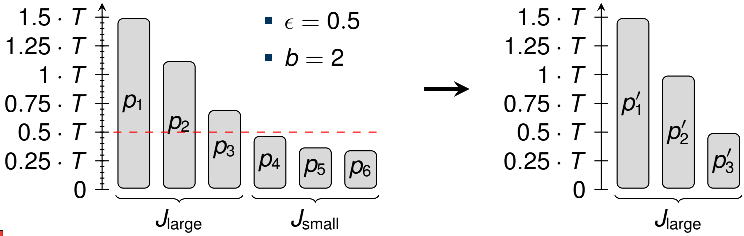
Can assume there are no jobs with $p_j \geq T$!



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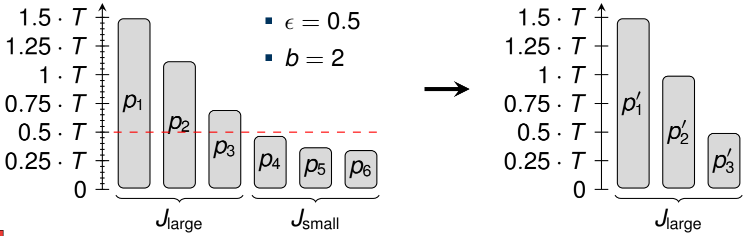

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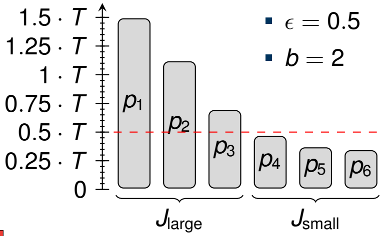
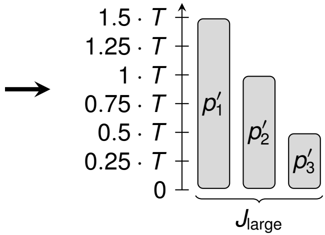
Assignments to one machine with makespan $\leq T$.



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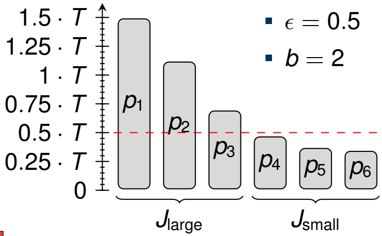

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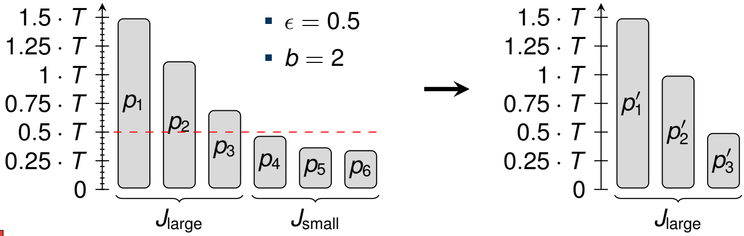

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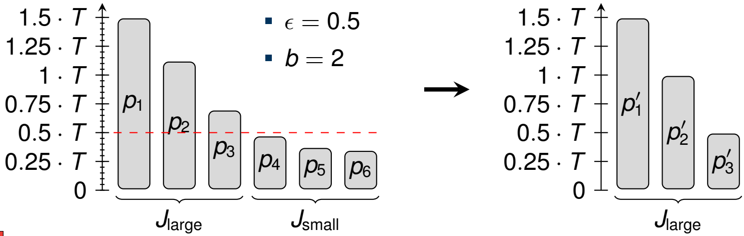

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Assign some jobs to one machine, and then use as few machines as possible for the rest.

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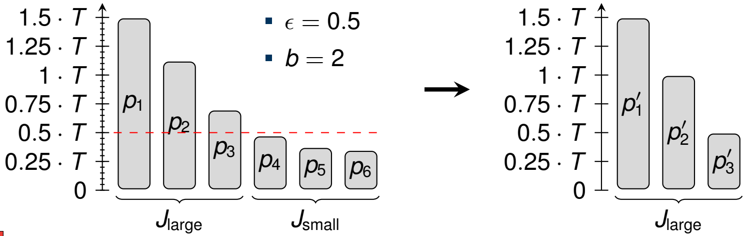

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Because for sufficiently small approximation ratio $1 + \epsilon$, the computed solution has to be optimal, and Parallel Machine Scheduling is strongly NP-hard.