Advanced Algorithms

I. Course Intro and Sorting Networks

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Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

(Tentative) List of Topics

IA Algorithms → IB Complexity Theory → II Advanced Algorithms

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- I. Sorting Networks (Sorting, Counting)
- II. Matrix Multiplication (and Parallel Algorithms)
- III. Linear Programming
- IV. Approximation Algorithms: Covering Problems
- V. Approximation Algorithms via Exact Algorithms
- VI. Approximation Algorithms: Travelling Salesman Problem
- VII. Approximation Algorithms: Randomisation and Rounding

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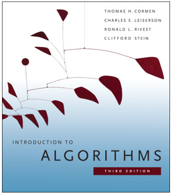

- closely follow CLRS3 and use the same numberring
- however, slides will be self-contained (mostly)

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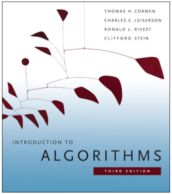

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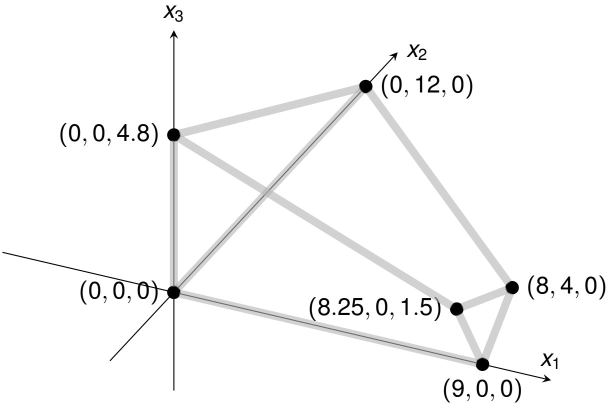
$$\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}$$

Linear Programming and Simplex



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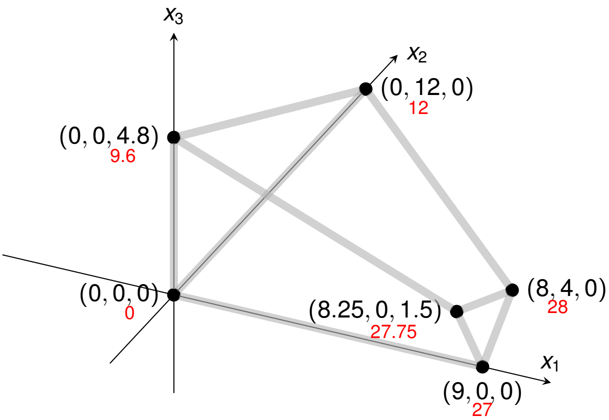

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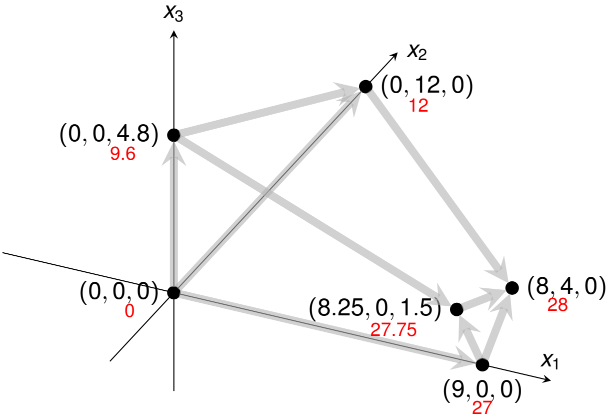

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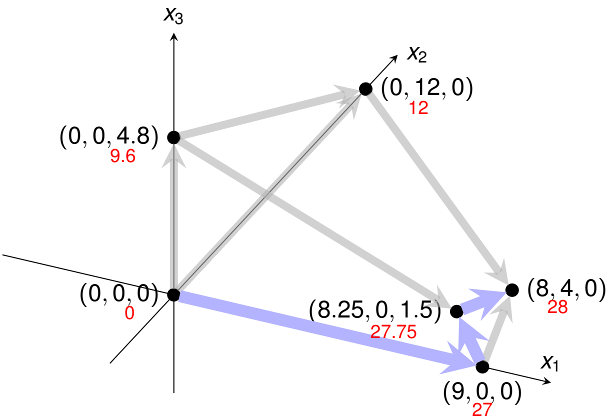

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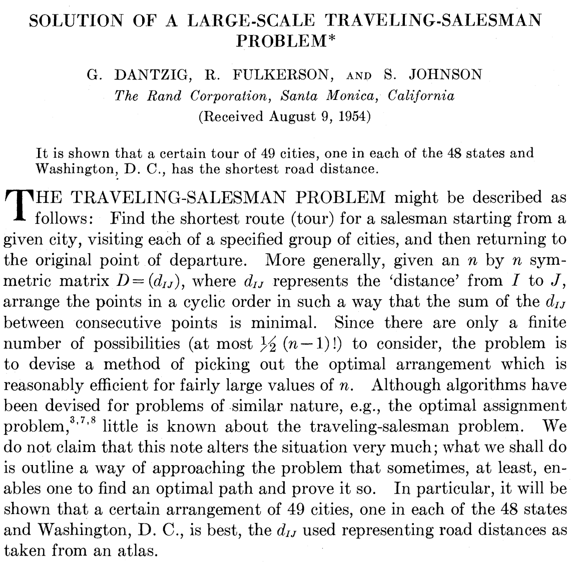
$$

The Original Article (1954)


SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON
The Rand Corporation, Santa Monica, California
(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an n by n symmetric matrix $D=(d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the d_{IJ} used representing road distances as taken from an atlas.

Travelling Salesman Problem: The 42 (49) Cities

1. Manchester, N. H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5. Charleston, W. Va.
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill.
9. Milwaukee, Wis.
10. Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13. Helena, Mont.
14. Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18. Carson City, Nev.
19. Los Angeles, Calif.
20. Phoenix, Ariz.
21. Santa Fe, N. M.
22. Denver, Colo.
23. Cheyenne, Wyo.
24. Omaha, Neb.
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28. Oklahoma City, Okla.
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32. Jackson, Miss.
33. New Orleans, La.
34. Birmingham, Ala.
35. Atlanta, Ga.
36. Jacksonville, Fla.
37. Columbia, S. C.
38. Raleigh, N. C.
39. Richmond, Va.
40. Washington, D. C.
41. Boston, Mass.
42. Portland, Me.

A. Baltimore, Md.
B. Wilmington, Del.
C. Philadelphia, Penn.
D. Newark, N. J.
E. New York, N. Y.
F. Hartford, Conn.
G. Providence, R. I.

Road Distances

TABLE I

ROAD DISTANCES BETWEEN CITIES IN ADJUSTED UNITS

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
2	8																																								
3	39	45																																							
4	37	47	9																																						
5	50	49	21	15																																					
6	61	62	21	20	17																																				
7	58	60	16	17	18	6																																			
8	59	60	15	20	26	17	10																																		
9	62	66	20	25	31	22	15	5																																	
10	81	81	40	44	50	41	35	24	20																																
11	103	107	62	67	72	63	57	46	41	23																															
12	108	117	66	71	77	68	61	51	46	26	11																														
13	145	149	104	108	114	106	99	88	84	63	49	40																													
14	181	185	140	144	150	142	135	124	120	99	85	76	35																												
15	187	191	146	150	156	142	137	130	125	105	90	81	41	10																											
16	161	170	120	124	130	115	110	104	105	90	72	64	34	31	27																										
17	142	146	101	104	111	97	91	85	86	75	51	59	29	53	48	21																									
18	174	178	133	138	143	129	123	117	118	107	83	84	54	46	35	26	31																								
19	185	186	142	143	140	130	126	124	128	118	93	101	72	69	58	58	43	26																							
20	164	165	120	123	124	106	106	105	110	104	86	97	71	93	82	62	42	45	22																						
21	137	139	94	96	94	80	78	77	84	77	56	64	65	90	87	58	36	68	50	30																					
22	117	122	77	80	83	68	62	60	61	50	34	42	49	82	77	60	30	62	70	49	21																				
23	114	118	73	78	84	69	63	57	59	48	28	36	43	77	72	45	27	59	69	55	27	5																			
24	85	89	44	48	53	41	34	28	29	22	23	35	69	105	102	74	56	88	99	81	54	32	29																		
25	77	80	36	40	46	34	27	19	21	14	29	40	77	114	111	84	64	96	107	87	60	40	37	8																	
26	87	89	44	46	46	30	28	29	32	27	36	47	78	116	112	84	66	98	95	75	47	36	39	12	11																
27	91	93	48	50	48	34	32	33	36	30	34	45	77	115	110	83	63	97	91	72	44	32	36	9	15	3															
28	105	106	62	63	64	47	46	49	54	48	46	59	85	119	115	88	66	98	79	59	31	36	42	28	33	21	20														
29	111	113	69	71	66	51	53	56	61	57	59	71	96	130	126	98	75	98	85	62	38	47	53	39	42	29	30	12													
30	91	92	50	51	46	30	34	38	43	49	60	71	103	141	136	109	90	115	99	81	53	61	62	36	34	24	28	20	20												
31	83	85	42	43	38	22	26	32	36	51	63	75	106	142	140	112	93	126	108	88	60	64	66	39	36	27	31	28	28	8											
32	89	91	55	55	50	34	39	44	49	63	76	87	120	155	150	123	100	123	109	86	62	71	78	52	49	39	44	35	24	15	12										
33	95	97	64	63	56	42	49	56	60	75	86	97	126	160	155	128	104	128	113	90	67	76	82	62	59	49	53	40	29	25	23	11									
34	74	81	44	43	35	23	30	39	44	62	78	89	121	159	155	127	108	136	124	101	75	79	81	54	50	42	46	43	39	23	14	14	21								
35	67	69	42	41	31	25	32	41	46	64	83	90	130	164	160	133	114	146	134	111	85	84	86	59	52	47	51	53	49	32	24	24	30	9							
36	74	76	61	60	42	44	51	60	66	83	102	110	147	185	179	155	133	159	146	122	98	105	107	79	71	66	70	70	60	48	40	36	33	25	18						
37	57	59	46	41	25	30	36	47	52	71	93	98	136	172	172	148	126	158	147	124	121	97	99	71	65	59	63	67	62	46	38	37	43	23	13	17					
38	45	46	41	34	20	34	38	48	53	73	96	99	137	176	178	151	131	163	159	135	108	102	103	73	67	64	69	75	72	54	46	49	54	34	24	29	12				
39	35	37	35	26	18	34	36	46	51	70	93	97	134	171	176	151	129	161	163	139	118	102	101	71	65	65	70	84	78	58	50	56	62	41	32	38	21	9			
40	29	33	30	21	18	35	33	40	45	65	87	91	117	166	171	144	125	157	156	139	113	95	97	67	60	62	67	79	82	62	53	59	66	45	38	45	27	15	6		
41	3	11	41	37	47	57	55	58	63	83	105	109	147	186	188	164	144	176	182	161	134	119	116	86	78	84	88	101	108	88	80	86	92	71	64	71	54	41	32	25	
42	5	12	55	41	53	64	61	61	66	84	111	113	150	186	192	166	147	180	188	167	140	124	119	90	87	90	94	107	114	77	86	92	98	80	74	77	60	48	38	32	6

The (Unique) Optimal Tour (699 Units $\approx$ 12,345 miles)

406

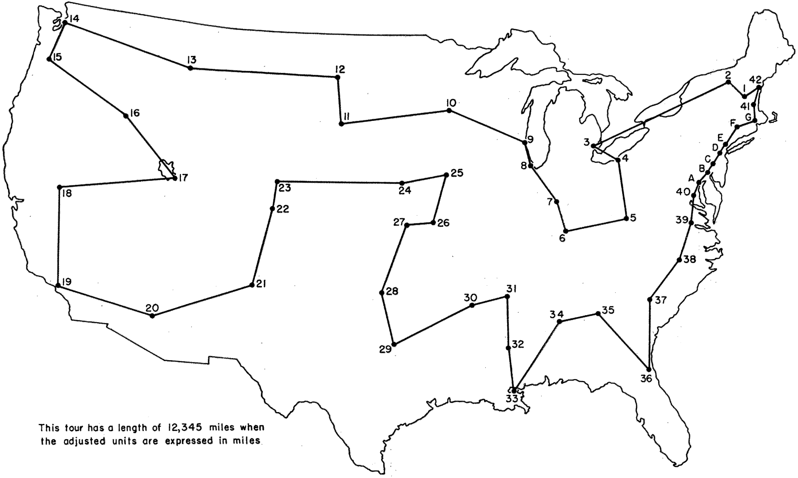


FIG. 16. The optimal tour of 49 cities.

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Overview: Sorting Networks

(Serial) Sorting Algorithms

- we already know several (comparison-based) sorting algorithms: Insertion sort, Bubble sort, Merge sort, Quick sort, Heap sort
- execute one operation at a time
- can handle arbitrarily large inputs
- sequence of comparisons is not set in advance

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Simple concept, but surprisingly deep and complex theory!

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  - comparator is a device with, on given two inputs, x and y, returns two outputs $x' = \min(x, y)$ and $y' = \max(x, y)$

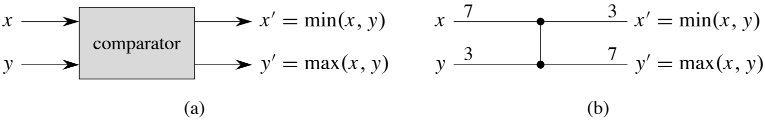

Figure 27.1 **(a)** A comparator with inputs x and y and outputs x' and y'. **(b)** The same comparator, drawn as a single vertical line. Inputs $x = 7$, $y = 3$ and outputs $x' = 3$, $y' = 7$ are shown.

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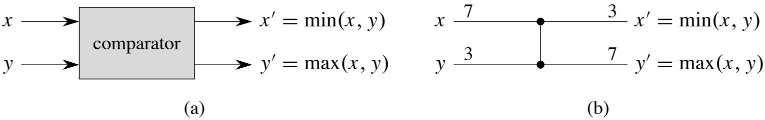
operates in $O(1)$



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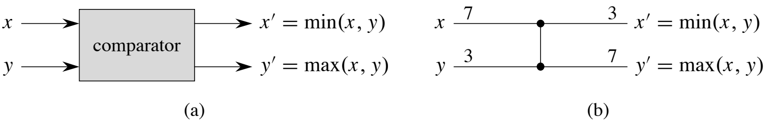

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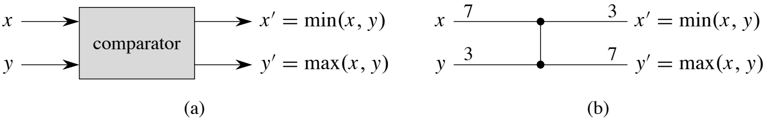
  - special wires: n input wires $a_1, a_2, \ldots, a_n$ and n output wires $b_1, b_2, \ldots, b_n$



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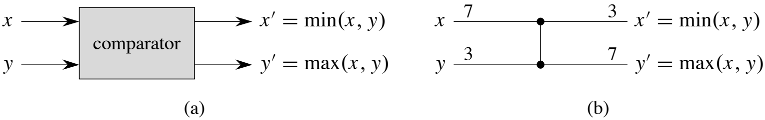
Convention: use the same name for both a wire and its value.



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A sorting network is a comparison network which works correctly (that is, it sorts every input)

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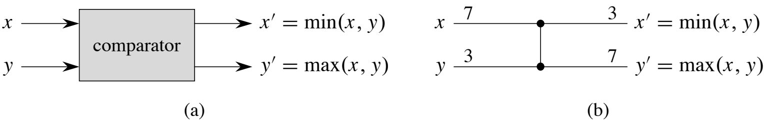

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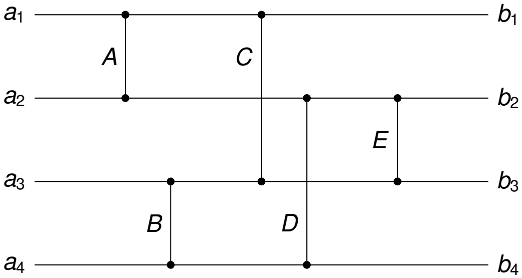

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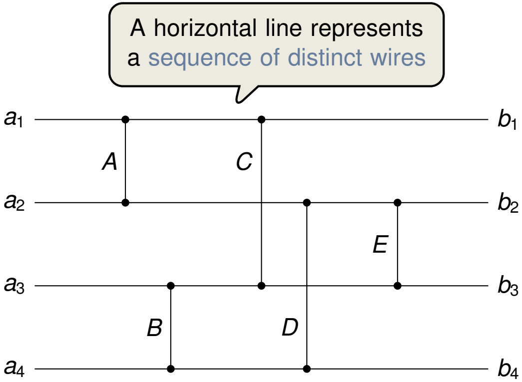

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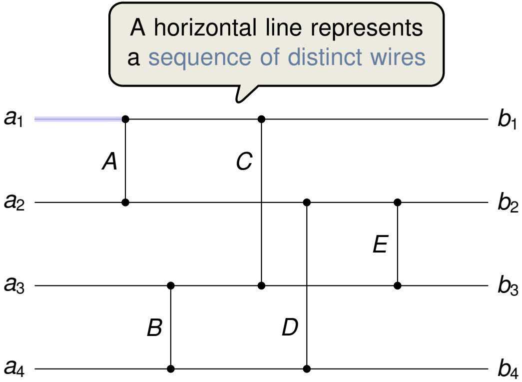

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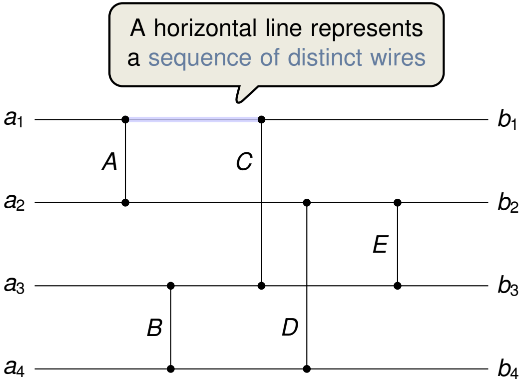

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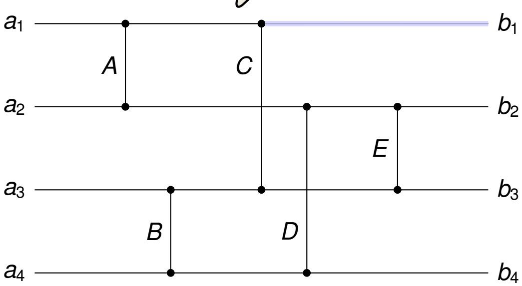
A horizontal line represents
a sequence of distinct wires



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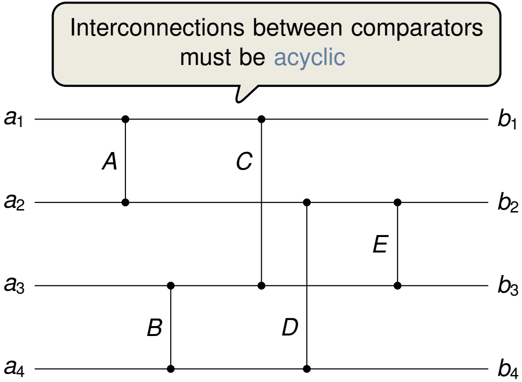

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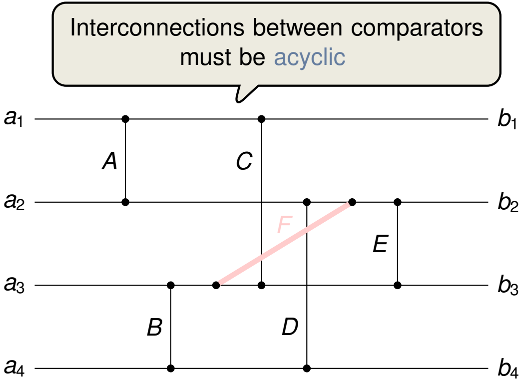
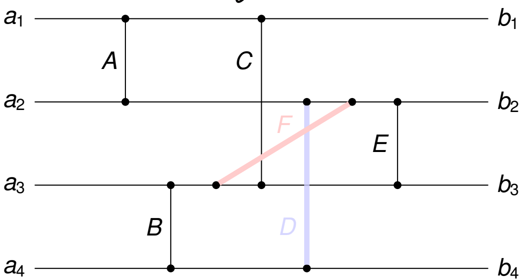


Example of a Comparison Network (Figure 27.2)

Interconnections between comparators must be acyclic



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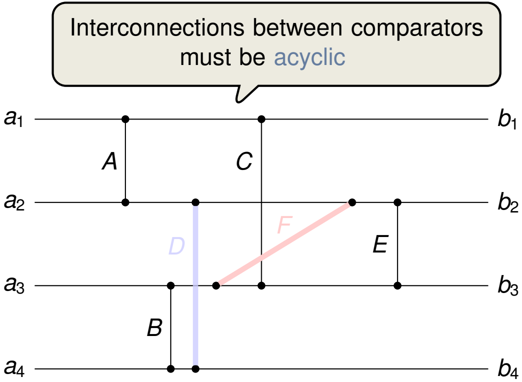

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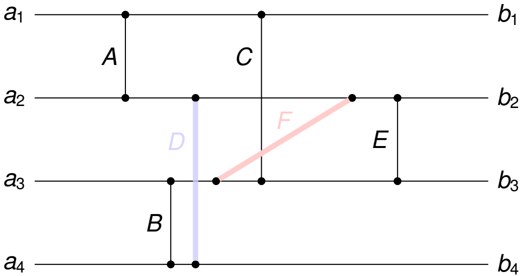

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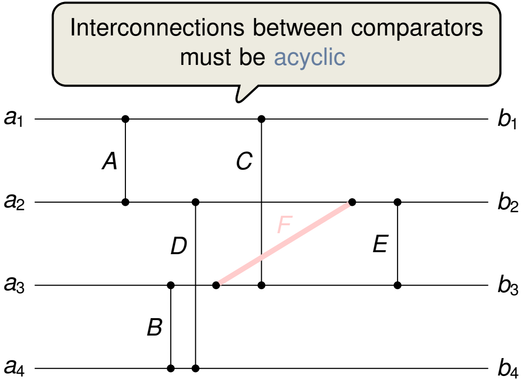

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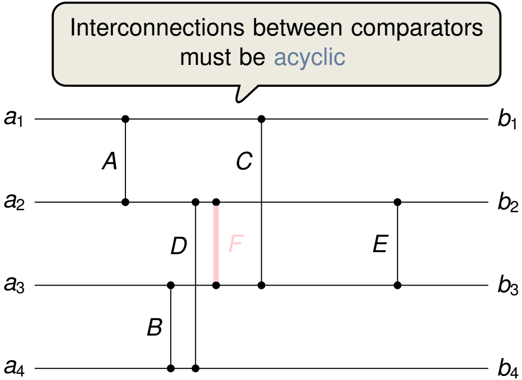

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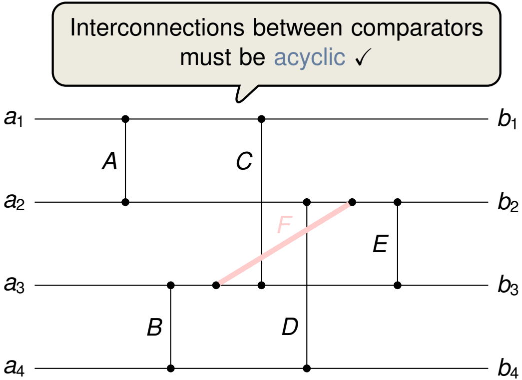

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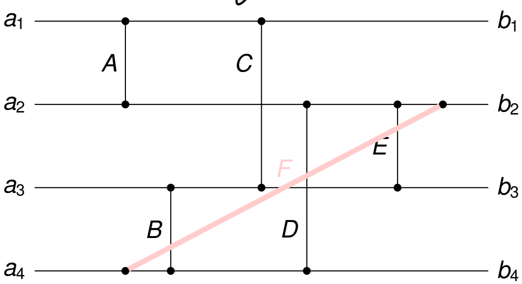

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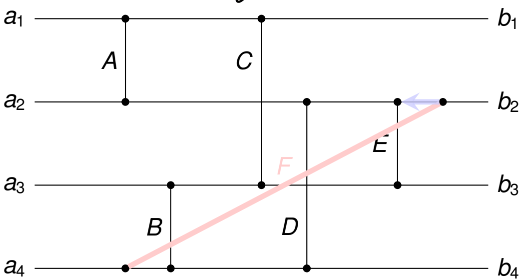

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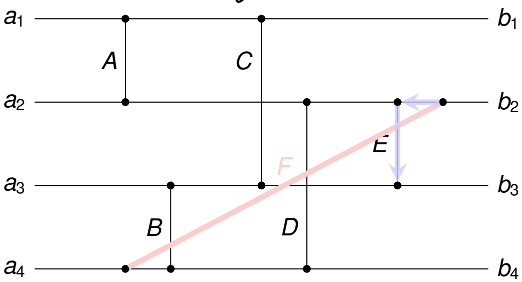

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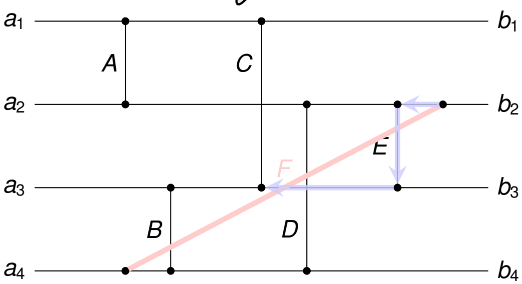

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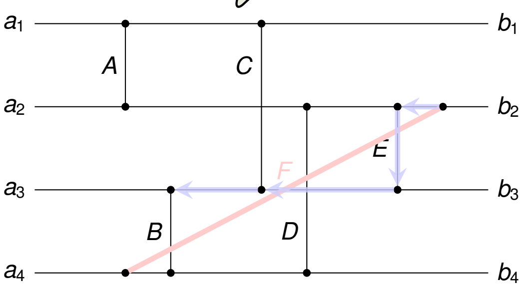

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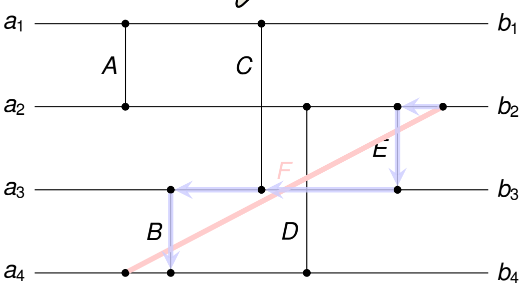

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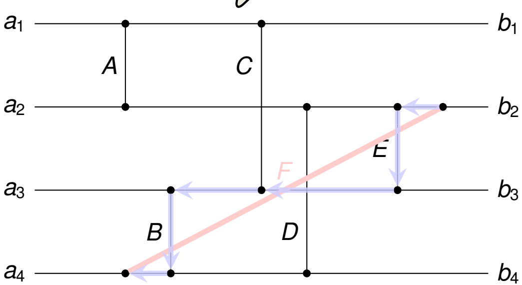

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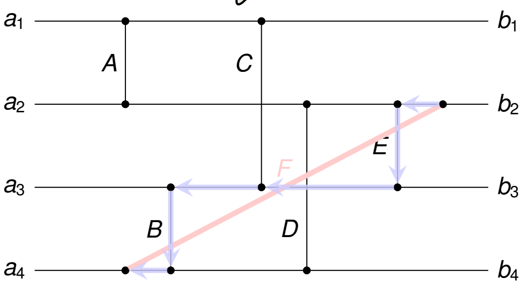
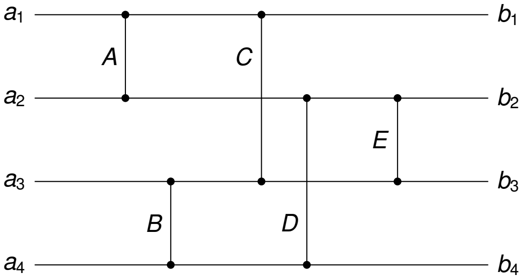


Tracing back a path must never cycle back on itself and go through the same comparator twice.

Example of a Comparison Network (Figure 27.2)



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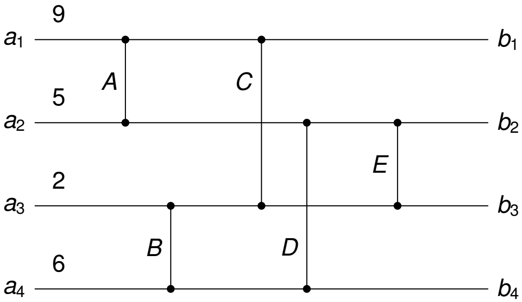

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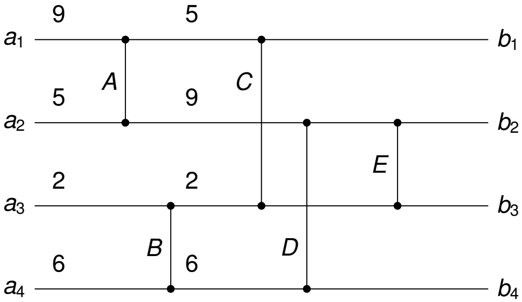

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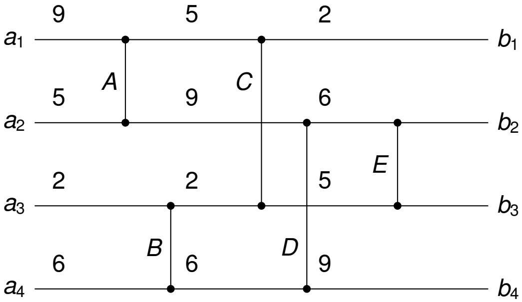

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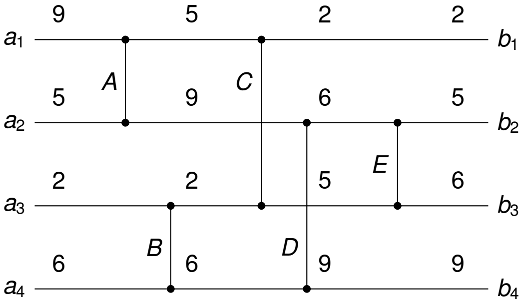
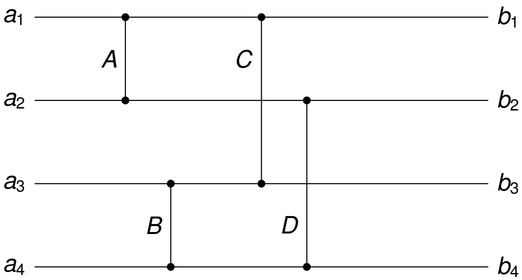


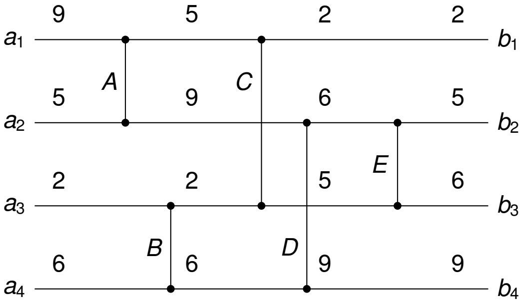
This network is in fact a sorting network (Exercise)

Example of a Comparison Network (Figure 27.2)



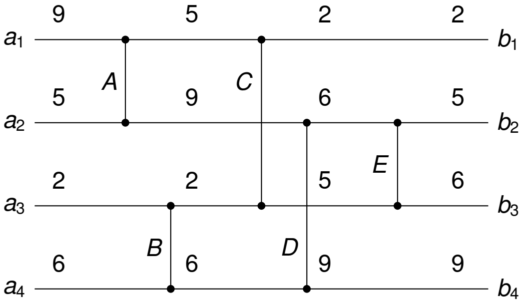
This network would not be a sorting network (Why??)

Example of a Comparison Network (Figure 27.2)



Depth of a wire:

Example of a Comparison Network (Figure 27.2)



Depth of a wire:

- Input wire has depth 0

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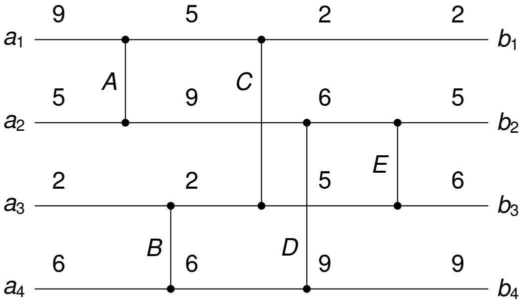
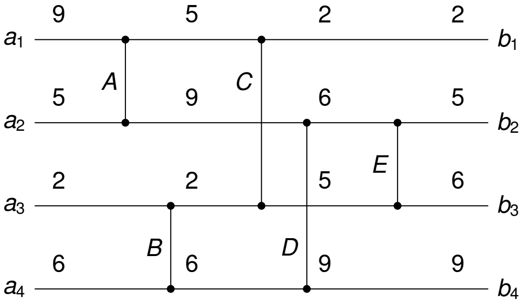


Depth of a wire:

- Input wire has depth 0
- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

Example of a Comparison Network (Figure 27.2)



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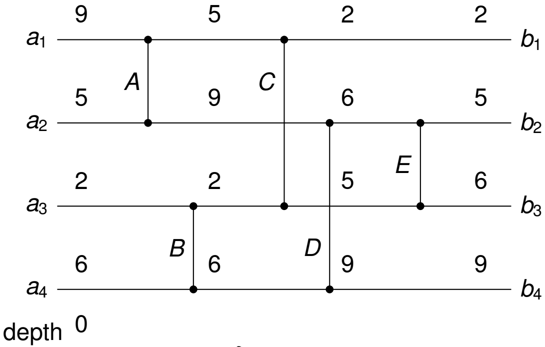

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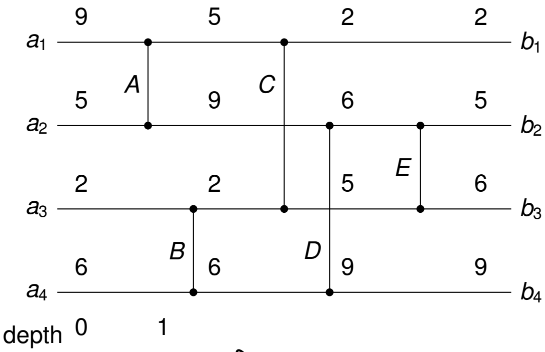

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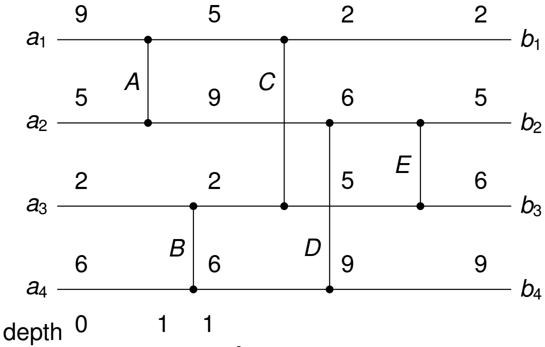

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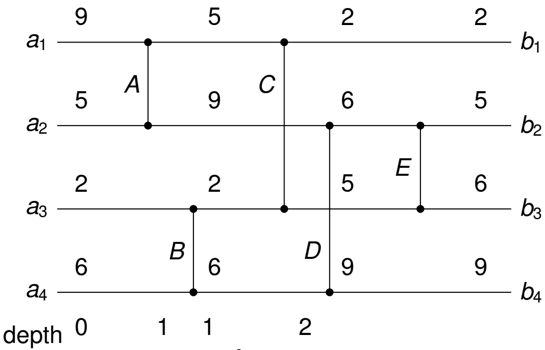

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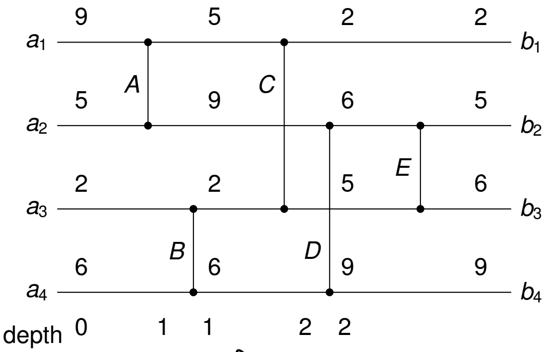

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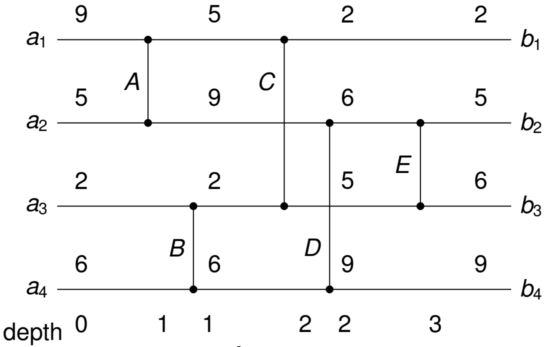

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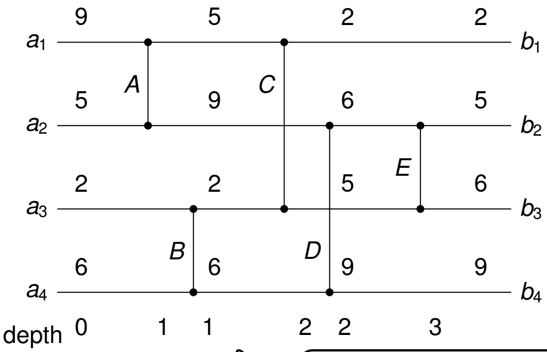

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Maximum depth of an output wire equals total running time

Zero-One Principle

Zero-One Principle: A sorting networks works correctly on arbitrary inputs if it works correctly on binary inputs.

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Lemma 27.1

If a comparison network transforms the input $a = \langle a_1, a_2, \ldots, a_n \rangle$ into the output $b = \langle b_1, b_2, \ldots, b_n \rangle$, then for any monotonically increasing function f, the network transforms $f(a) = \langle f(a_1), f(a_2), \ldots, f(a_n) \rangle$ into $f(b) = \langle f(b_1), f(b_2), \ldots, f(b_n) \rangle$.

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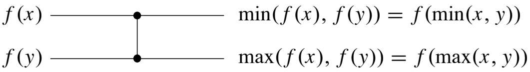

$$\begin{tikzcd} a \arrow[r, "\text{Network}"] \arrow[d, "f"'] & b \arrow[d, "f"] \\ f(a) \arrow[r, "\text{Network}"] & f(b) \end{tikzcd}$$

Figure 27.4 The operation of the comparator in the proof of Lemma 27.1. The function f is monotonically increasing.

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Theorem 27.2 (Zero-One Principle)

If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

Proof of the Zero-One Principle

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$$f(x) = \begin{cases} 0 & \text{if } x \leq a_i, \\ 1 & \text{if } x > a_i. \end{cases}$$

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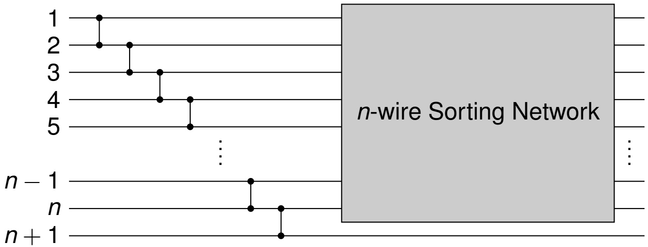
- Since the network places a_j before a_i, by the previous lemma $\Rightarrow f(a_j)$ is placed before $f(a_i)$
- But $f(a_j) = 1$ and $f(a_i) = 0$, which contradicts the assumption that the network sorts all sequences of 0's and 1's correctly $\square$

Some Basic (Recursive) Sorting Networks



???

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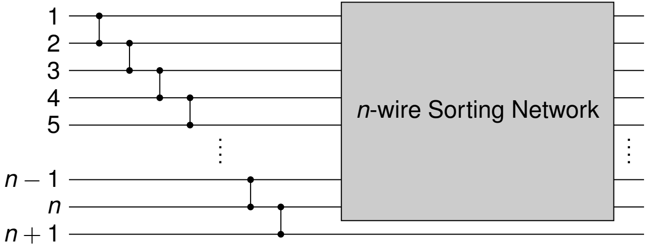

Bubble Sort

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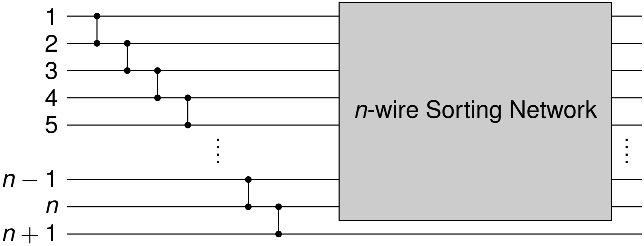

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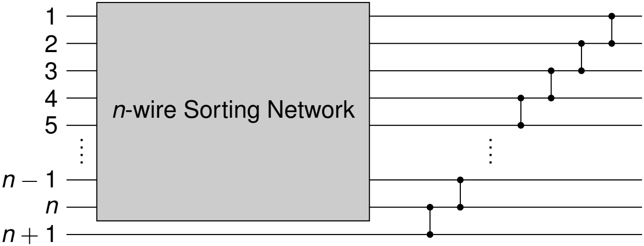

???

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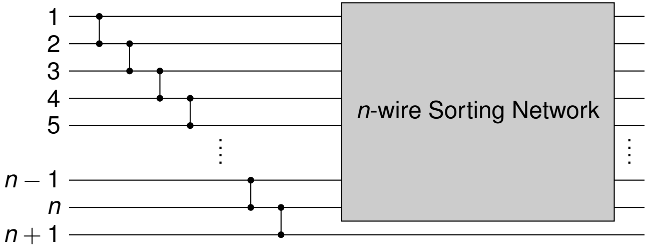

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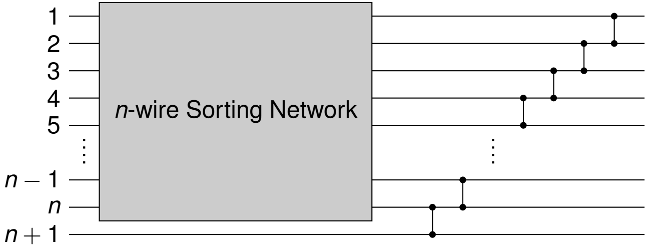
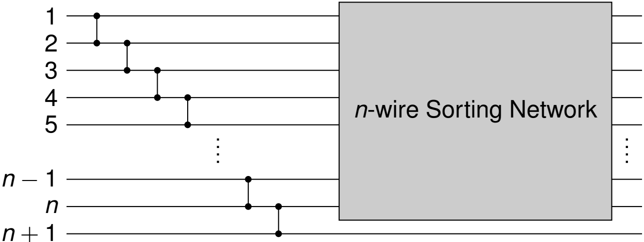


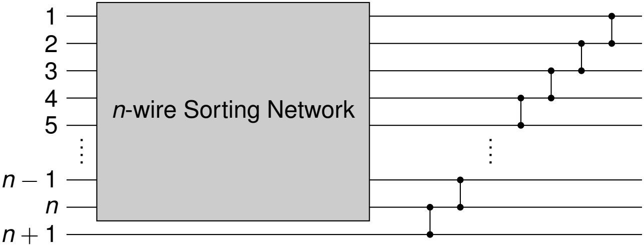
Insertion Sort

Some Basic (Recursive) Sorting Networks



Bubble Sort

These are Sorting Networks, but with depth $\Theta(n)$.



Insertion Sort

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Bitonic Sequences

Bitonic Sequence

A sequence is bitonic if it monotonically increases and then monotonically decreases, or can be circularly shifted to become monotonically increasing and then monotonically decreasing.

Sequences of one or two numbers are defined to be bitonic.

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Examples:

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- $\langle 6, 9, 4, 2, 3, 5 \rangle$?

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- $\langle 9, 8, 3, 2, 4, 6 \rangle$?

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- $\langle 9, 8, 3, 2, 4, 6 \rangle$ ✓
- $\cancel{\langle 4, 5, 7, 1, 2, 6 \rangle}$
- binary sequences: $0^i 1^j 0^k$, or, $1^i 0^j 1^k$, for $i, j, k \geq 0$.

Towards Bitonic Sorting Networks

Half-Cleaner

A half-cleaner is a comparison network of depth 1 in which input wire i is compared with wire $i + n/2$ for $i = 1, 2, \ldots, n/2$.

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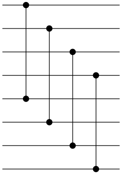

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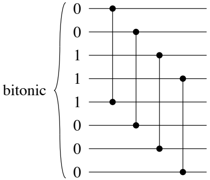

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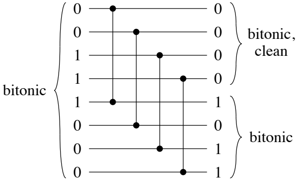

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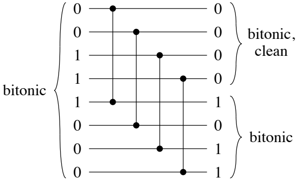

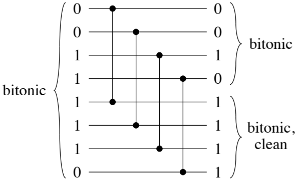

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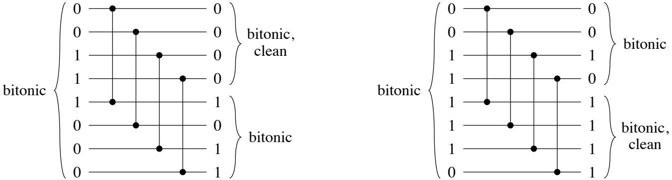
Lemma 27.3

If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

- both the top half and the bottom half are bitonic,
- every element in the top is not larger than any element in the bottom,
- at least one half is clean.



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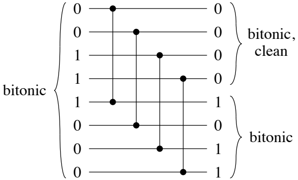

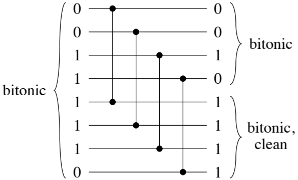

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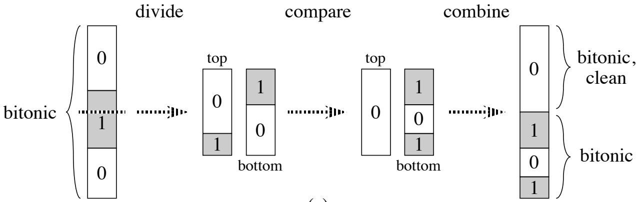

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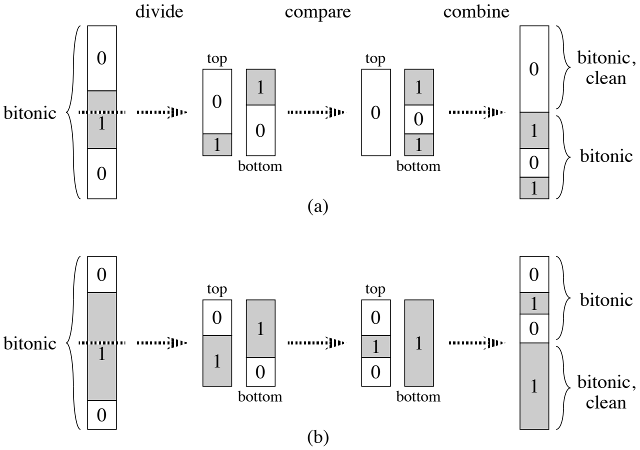

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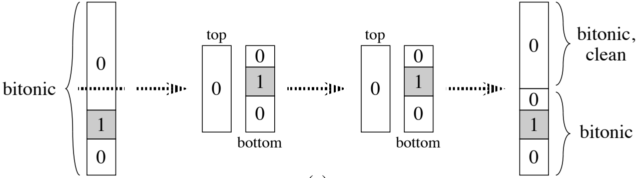

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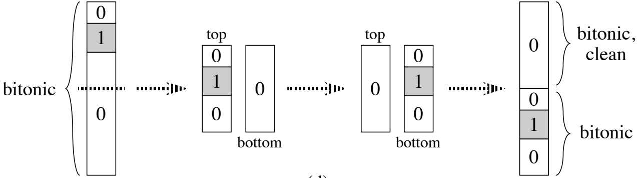

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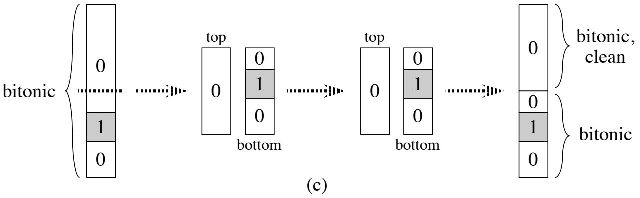

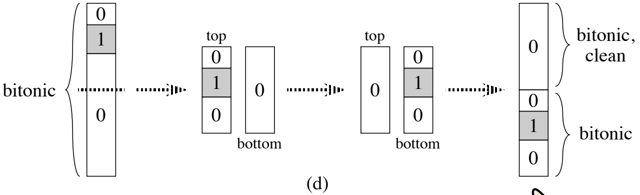

This suggests a recursive approach, since it now suffices to sort the top and bottom half separately.

The Bitonic Sorter

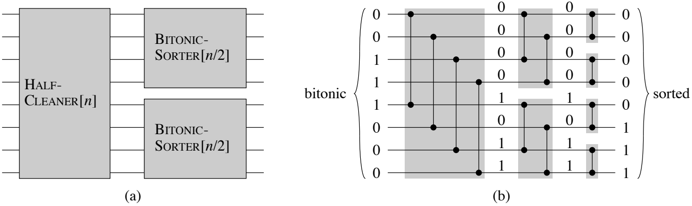

Figure 27.9 The comparison network BITONIC-SORTER[n], shown here for $n = 8$. **(a)** The recursive construction: HALF-CLEANER[n] followed by two copies of BITONIC-SORTER[$n/2$] that operate in parallel. **(b)** The network after unrolling the recursion. Each half-cleaner is shaded. Sample zero-one values are shown on the wires.

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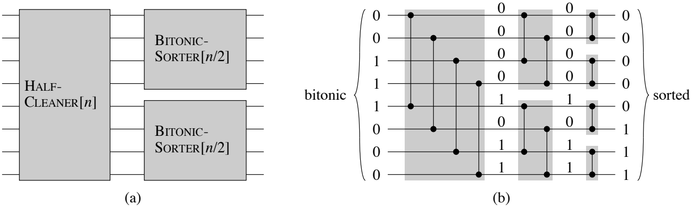

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Recursive Formula for depth $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + 1 & \text{if } n = 2^k. \end{cases}$$

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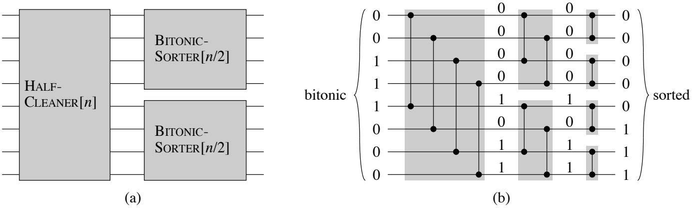

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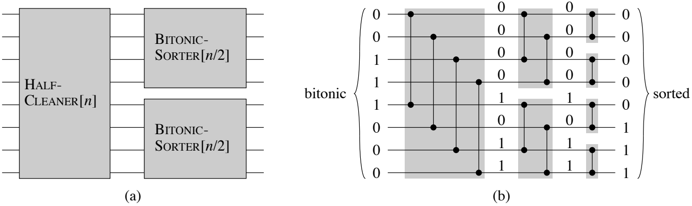

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BITONIC-SORTER[n] has depth $\log n$ and sorts any zero-one bitonic sequence.

Merging Networks

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- will be based on a modification of BITONIC-SORTER[n]

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Hence in order to merge the sequences X and Y, it suffices to perform a bitonic sort on X concatenated with Y^R.

Construction of a Merging Network (1/2)

- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$
- We know it suffices to bitonically sort $\langle a_1, a_2, \ldots, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+1} \rangle$
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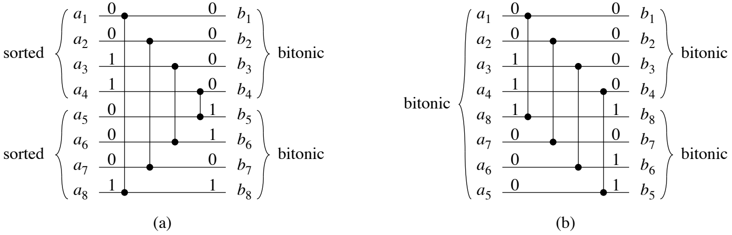

Figure 27.10 Comparing the first stage of MERGER$[n]$ with HALF-CLEANER$[n]$, for $n = 8$. **(a)** The first stage of MERGER$[n]$ transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2}\rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n\rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n\rangle$. **(b)** The equivalent operation for HALF-CLEANER$[n]$. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1}\rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1}\rangle$.

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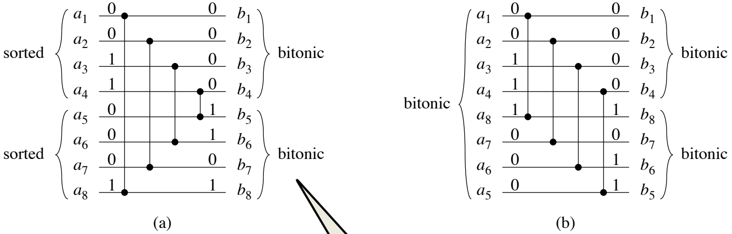

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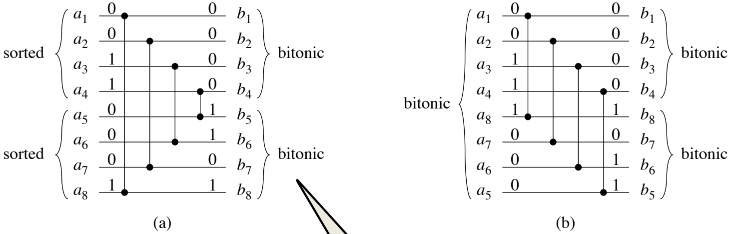
$\Rightarrow$ First part of MERGER[n] compares inputs i and $n-i+1$ for $i = 1, 2, \ldots, n/2$

- Remaining part is identical to BITONIC-SORTER[n]



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Construction of a Merging Network (2/2)

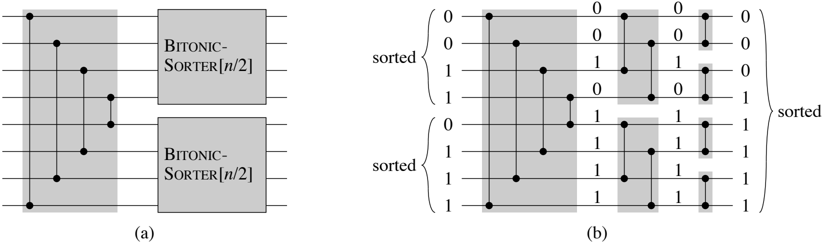


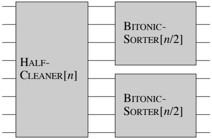
Figure 27.11 A network that merges two sorted input sequences into one sorted output sequence. The network MERGER[n] can be viewed as BITONIC-SORTER[n] with the first half-cleaner altered to compare inputs i and $n - i + 1$ for $i = 1, 2, \ldots, n/2$. Here, $n = 8$. **(a)** The network decomposed into the first stage followed by two parallel copies of BITONIC-SORTER[$n/2$]. **(b)** The same network with the recursion unrolled. Sample zero-one values are shown on the wires, and the stages are shaded.

Construction of a Sorting Network

Main Components

1. BITONIC-SORTER[n]
   - sorts any bitonic sequence
   - depth $\log n$

1.



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2. MERGER[n]
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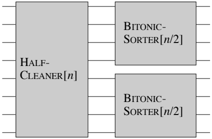
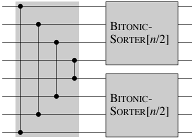


2.



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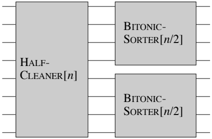

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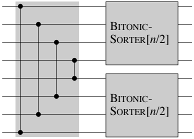
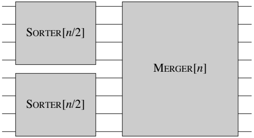


Batcher's Sorting Network

- SORTER[n] is defined recursively:
  - If $n = 2^k$, use two copies of SORTER[$n/2$] to sort two subsequences of length $n/2$ each. Then merge them using MERGER[n].
  - If $n = 1$, network consists of a single wire.



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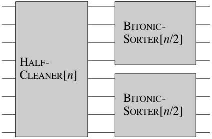
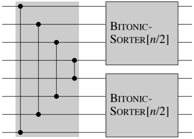
   - depth $\log n$
2. MERGER[n]
   - merges two sorted input sequences
   - depth $\log n$

1.



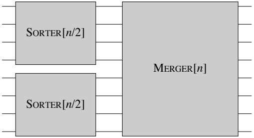
2.



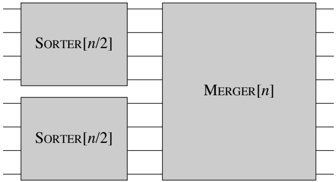
Batcher's Sorting Network

- SORTER[n] is defined recursively:
  - If $n = 2^k$, use two copies of SORTER[$n/2$] to sort two subsequences of length $n/2$ each. Then merge them using MERGER[n].
  - If $n = 1$, network consists of a single wire.

can be seen as a parallel version of merge sort



Unrolling the Recursion (Figure 27.12)



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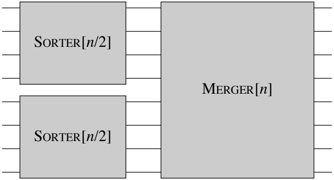
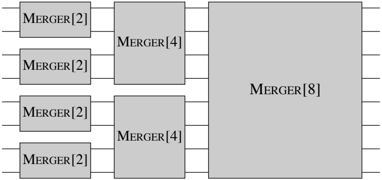
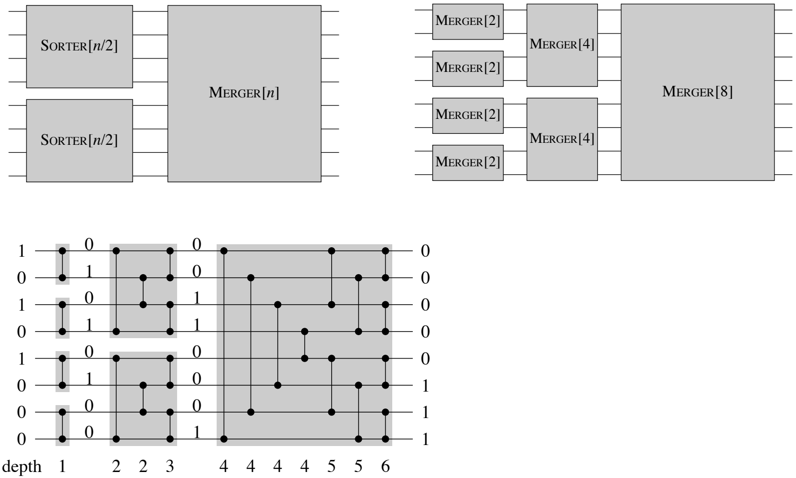
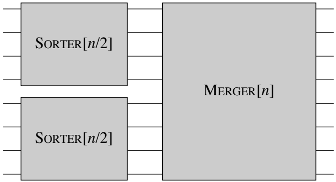


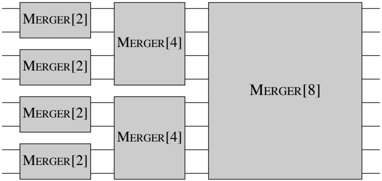


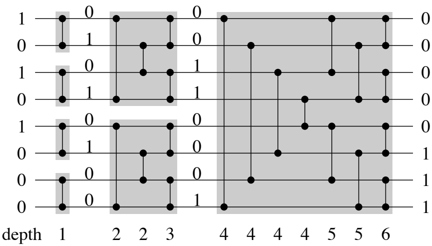


SORTER[n/2]
SORTER[n/2]
MERGER[n]
MERGER[2]
MERGER[2]
MERGER[2]
MERGER[2]
MERGER[4]
MERGER[4]
MERGER[8]
1 0 1 0 1 0 0 0
0 1 0 1 0 1 0 0
0 0 1 1 0 0 0 1
0 0 0 0 0 1 1 1
depth 1 2 2 3 4 4 4 4 5 5 6

Unrolling the Recursion (Figure 27.12)







Recursion for $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + \log n & \text{if } n = 2^k. \end{cases}$$

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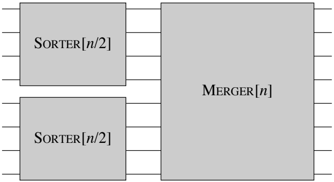

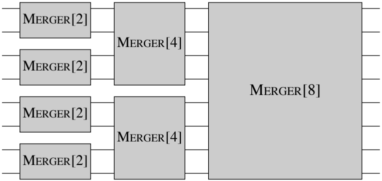

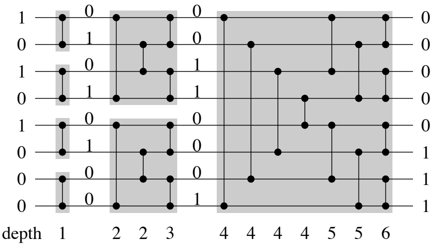

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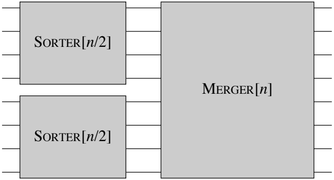

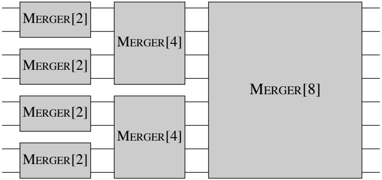

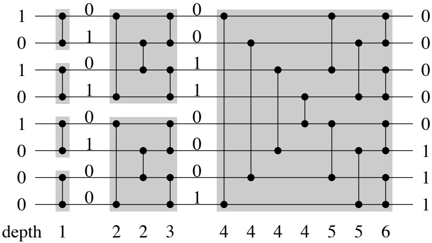

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SORTER[n] has depth $\Theta(\log^2 n)$ and sorts any input.

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $O(\log n)$.

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Quite elaborate construction, and involves huges constants.

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Perfect Halver

A perfect halver is a comparison network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

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Perfect halver of depth $\log n$ exist $\rightsquigarrow$ yields sorting networks of depth $\Theta((\log n)^2)$.

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An (n, ϵ)-approximate halver, $\epsilon < 1$, is a comparison network that for every $k = 1, 2, \ldots, n/2$ places at most ϵk of its k smallest keys in $b_{n/2+1}, \ldots, b_n$ and at most ϵk of its k largest keys in $b_1, \ldots, b_{n/2}$.

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We will prove that such networks can be constructed in constant depth!

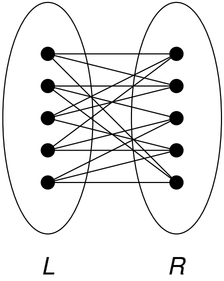
Expander Graphs

Expander Graphs

A bipartite (n, d, μ)-expander is a graph with:

- G has n vertices ($n/2$ on each side)
- the edge-set is union of d perfect matchings
- For every subset $S \subseteq V$ being in one part,

$$|N(S)| > \min\{\mu \cdot |S|, n/2 - |S|\}$$



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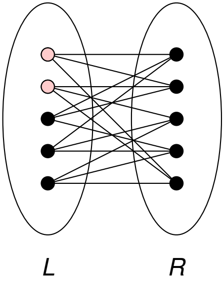

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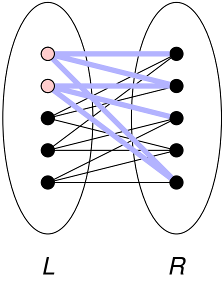

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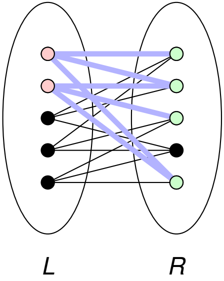

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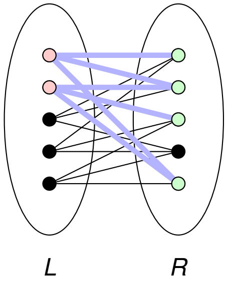
Specific definition tailored for sorting network - many other variants exist!



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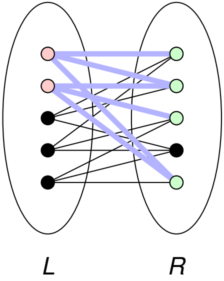
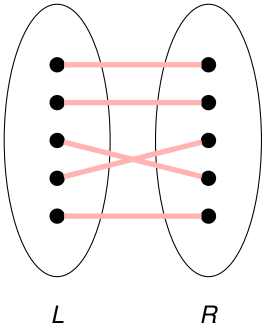


Expander Graphs:

- **probabilistic construction** "easy": take d (disjoint) random matchings
- **explicit construction** is a deep mathematical problem with ties to number theory, group theory, combinatorics etc.
- **many applications** in networking, complexity theory and coding theory

From Expanders to Approximate Halvers



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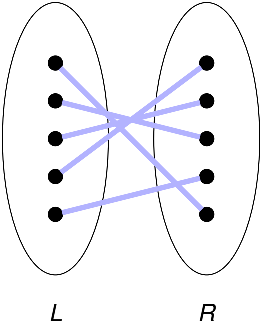

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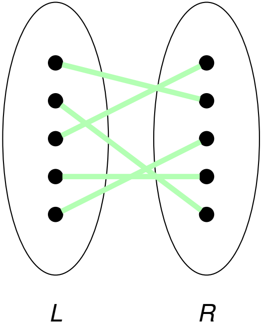

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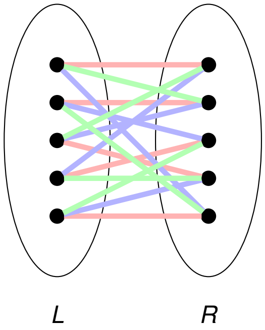

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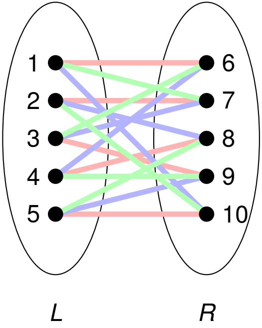

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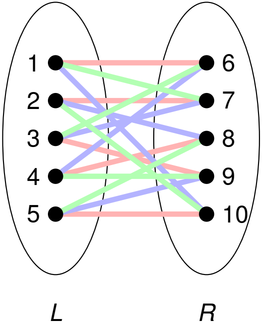

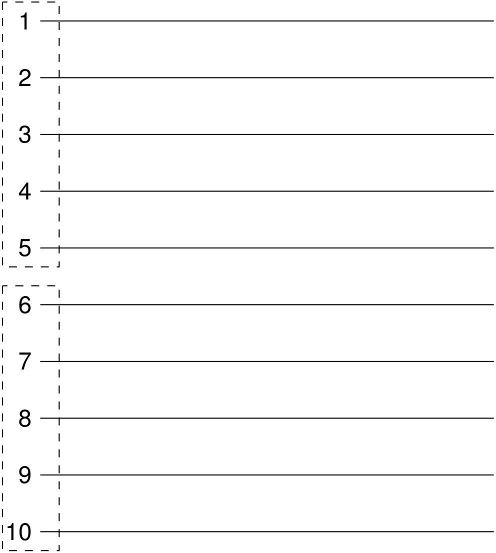

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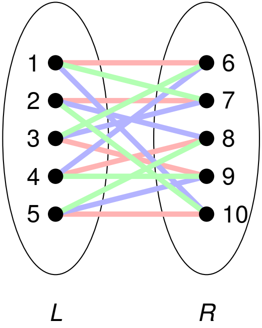

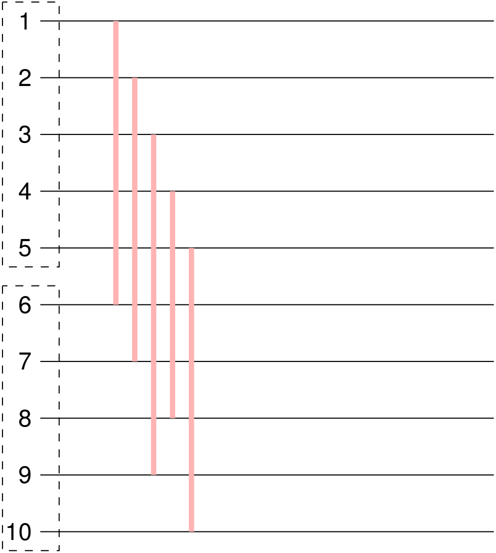

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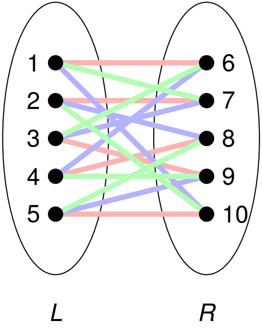

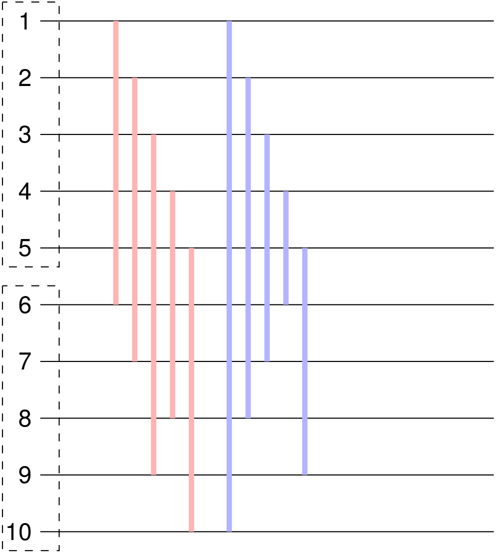

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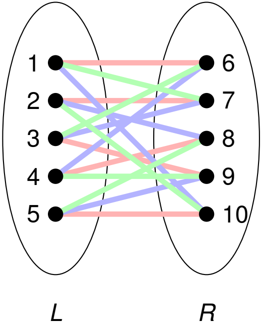
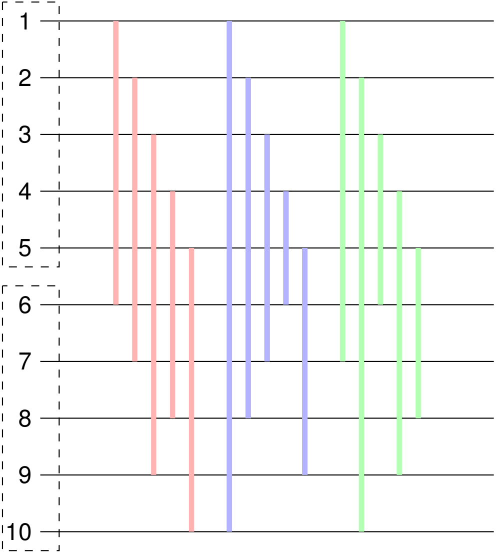
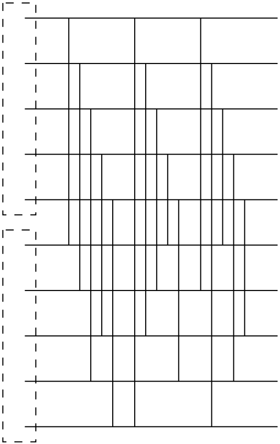




Existence of Approximate Halvers (non-examinable)

Proof:



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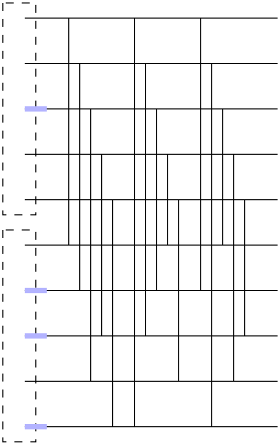
- $X :=$ keys with the k smallest inputs



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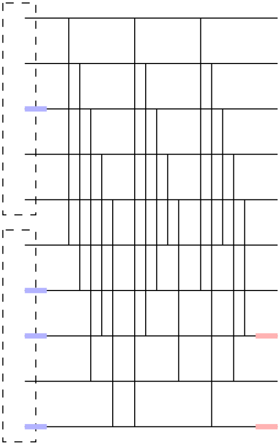

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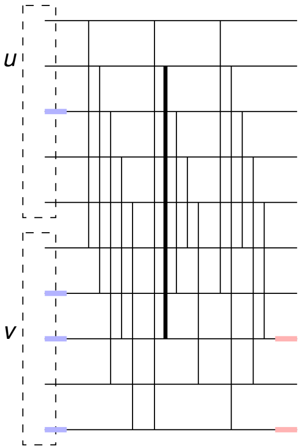

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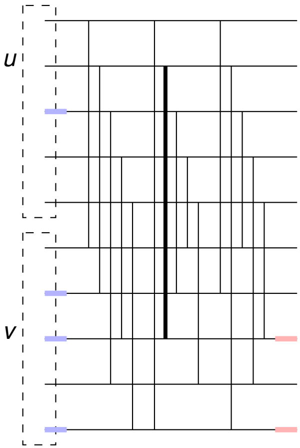

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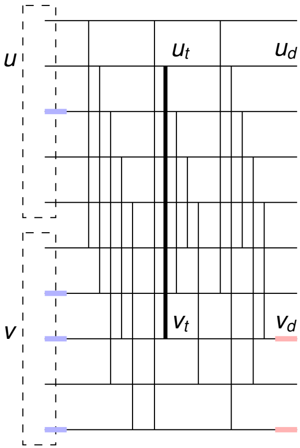

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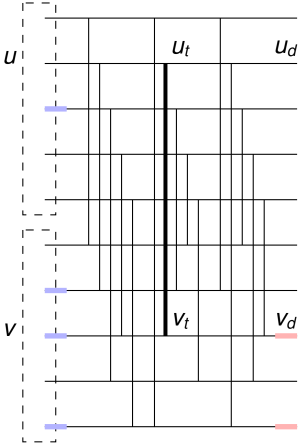

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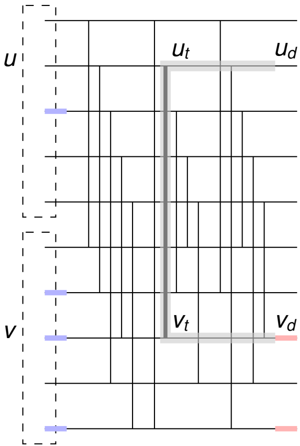

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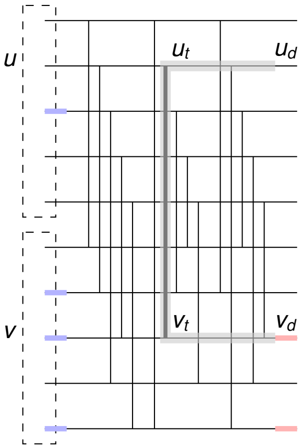
- Further: $u_d \leq u_t \leq v_t \leq v_d \Rightarrow u_d \in X$
- Since u was arbitrary:

$$|Y| + |N(Y)| \leq k.$$



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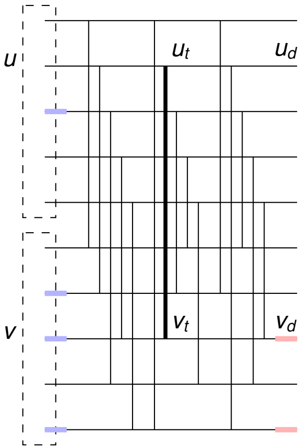

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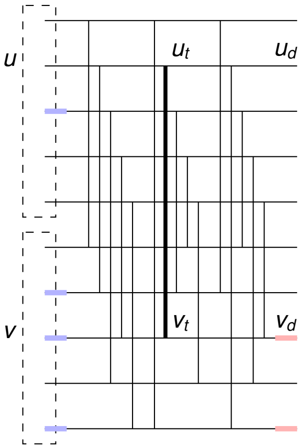

Existence of Approximate Halvers (non-examinable)

Proof:

- $X :=$ keys with the k smallest inputs
- $Y :=$ wires in lower half with k smallest outputs
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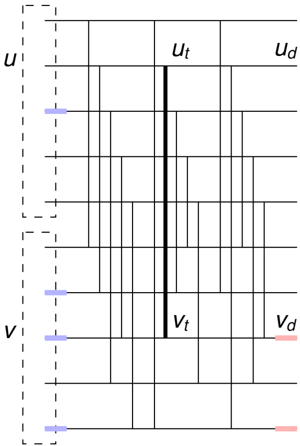

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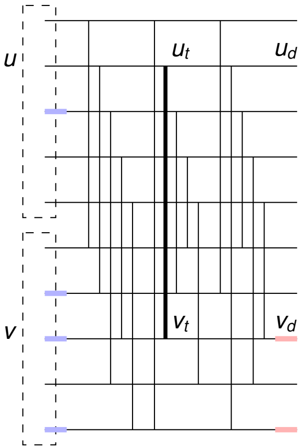

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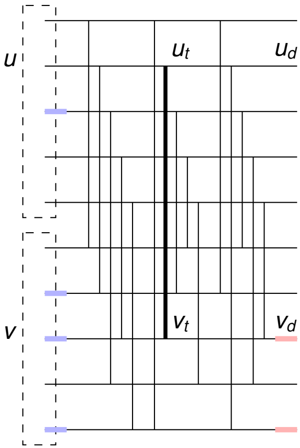

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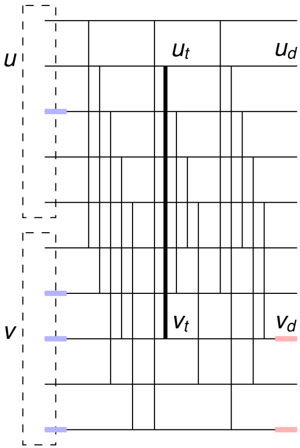
Here we used that $k \leq n/2$



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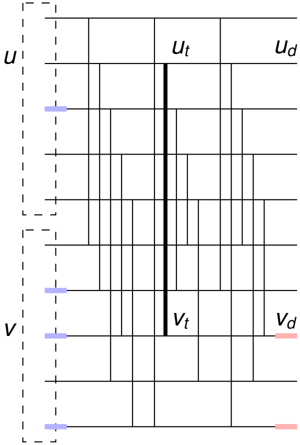
- Same argument $\Rightarrow$ at most $\epsilon \cdot k$, $\epsilon := 1/(\mu + 1)$, of the k largest input keys are placed in $b_1, \ldots, b_{n/2}$. □



- typical application of expander graphs in parallel algorithms
- Much more work needed to construct the AKS sorting network

AKS network vs. Batcher's network

Donald E. Knuth (Stanford)

"Batcher's method is much better, unless n exceeds the total memory capacity of all computers on earth!"

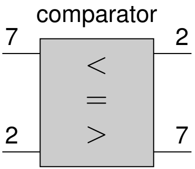
Richard J. Lipton (Georgia Tech)

*"The AKS sorting network is **galactic**: it needs that n be larger than 2^{78} or so to finally be smaller than Batcher's network for n items."*

Siblings of Sorting Network

Sorting Networks

- sorts any input of size *n*
- special case of Comparison Networks



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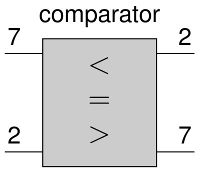
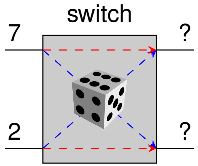


Switching (Shuffling) Networks

- creates a random permutation of n items
- special case of Permutation Networks



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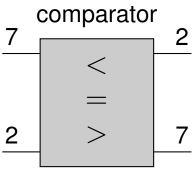

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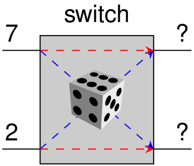
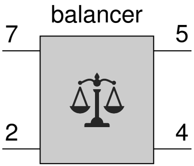


Counting Networks

- balances any stream of tokens over n wires
- special case of Balancing Networks



Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

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Distributed Counting

Processors collectively assign successive values from a given range.

Values could represent addresses in memories or destinations on an interconnection network

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

Balancing Networks

- constructed in a similar manner like sorting networks
- instead of comparators, consists of balancers
- balancers are asynchronous flip-flops that forward tokens from its inputs to one of its two outputs alternately (top, bottom, top,...)

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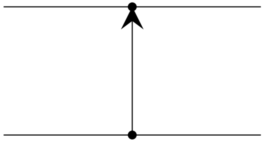

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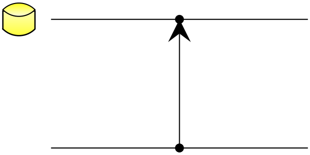

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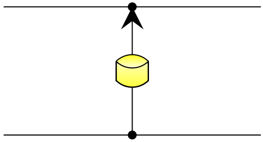

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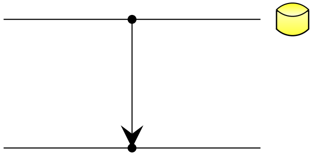

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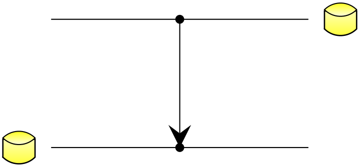

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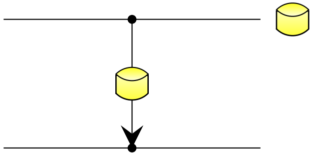

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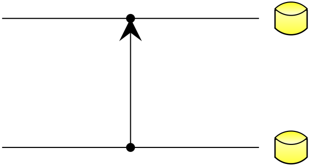

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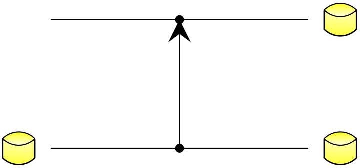

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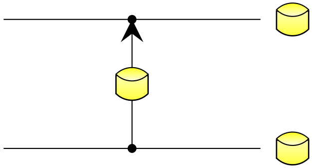

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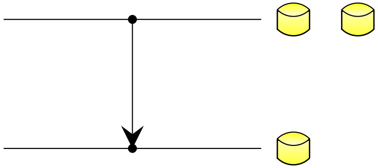

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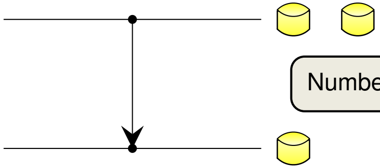

Number of tokens differs by at most one

Bitonic Counting Network

Counting Network (Formal Definition)

1. Let $x_1, x_2, \ldots, x_n$ be the number of tokens (ever received) on the designated input wires
2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires

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3. In a quiescent state: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$
4. A counting network is a balancing network with the **step-property:**

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Bitonic Counting Network: Take Batcher's Sorting Network and replace each comparator by a balancer.

Correctness of the Bitonic Counting Network (non-examinable)

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1$, then $\exists!\ j = 1, 2, \ldots, n$ with $x_j = y_j + 1$ and $x_i = y_i$ for $j \neq i$.

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Key Lemma

Consider a $\textsc{Merger}[n]$. Then if the inputs $x_1, \ldots, x_{n/2}$ and $x_{n/2+1}, \ldots, x_n$ have the step property, then so does the output $y_1, \ldots, y_n$.

Proof (by induction on n being a power of 2)

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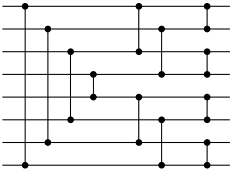

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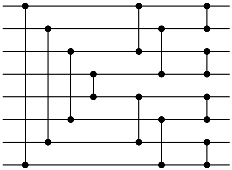

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- Case $n = 2$ is clear, since MERGER[2] is a single balancer

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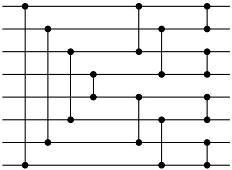

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
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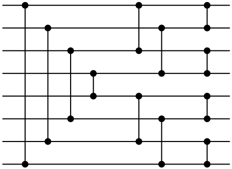

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- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks

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1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
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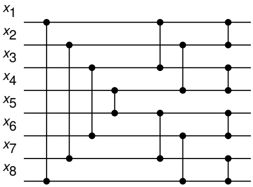

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks

Correctness of the Bitonic Counting Network (non-examinable)

Facts

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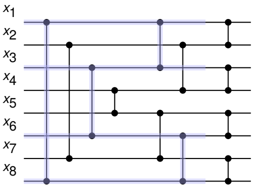

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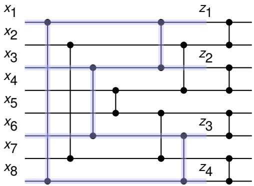

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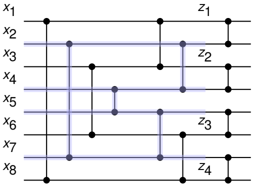

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Correctness of the Bitonic Counting Network (non-examinable)

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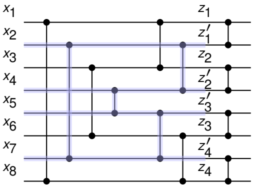

Proof (by induction on n being a power of 2)

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- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ have the step property

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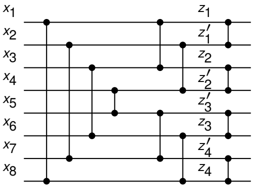

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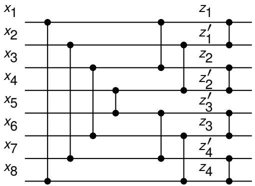

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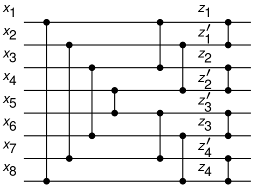

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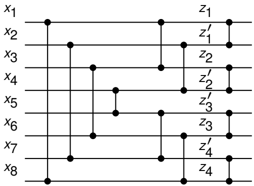

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- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓

Correctness of the Bitonic Counting Network (non-examinable)

Facts

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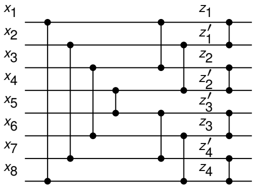

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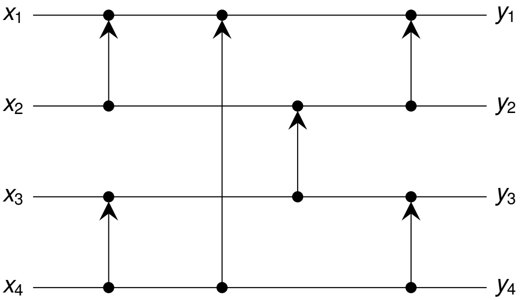
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓
- Case 2: If $|Z - Z'| = 1$, F3 implies $z_i = z'_i$ for $i = 1, \ldots, n/2$ except a unique j with $z_j \neq z'_j$.
  Balancer between z_j and z'_j will ensure that the step property holds.

Bitonic Counting Network in Action (Asychnronous Execution)



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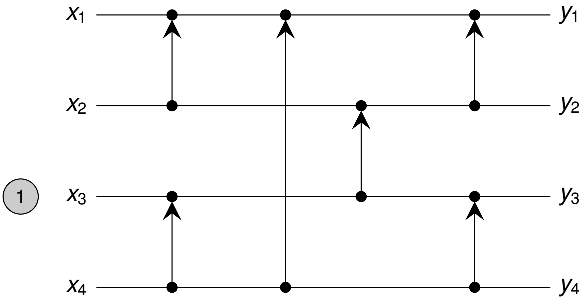

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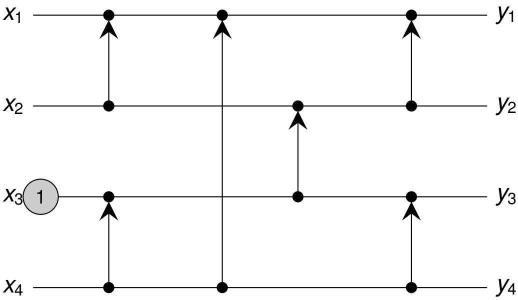

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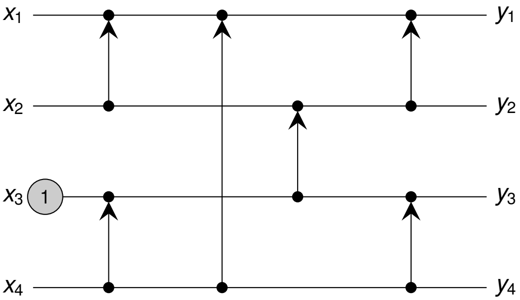

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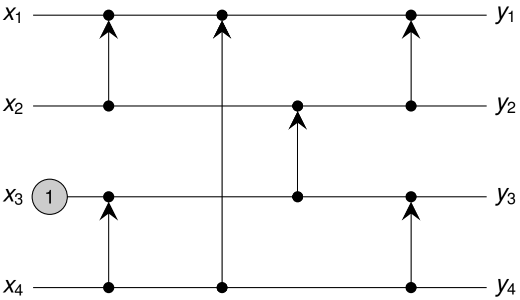

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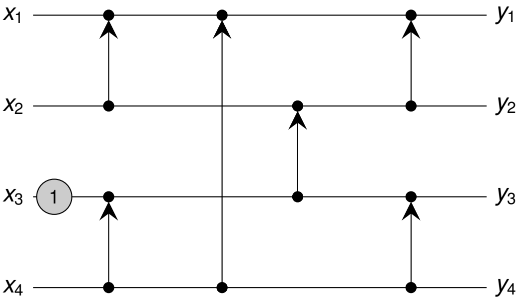

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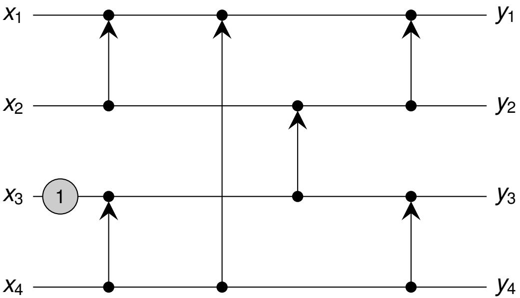

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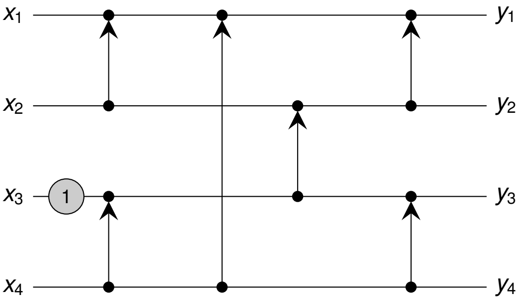

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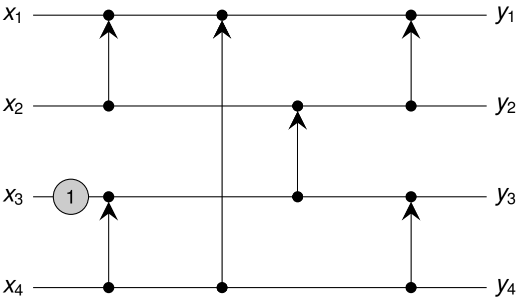

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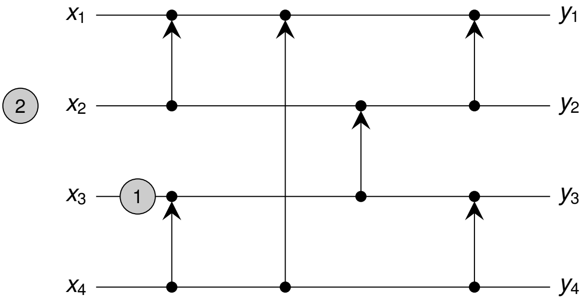

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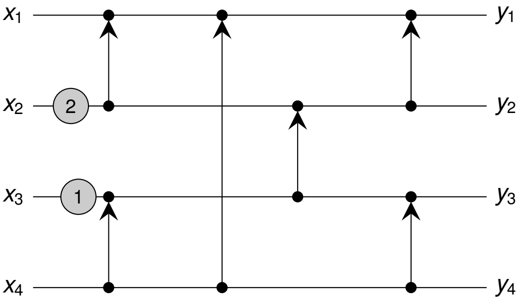

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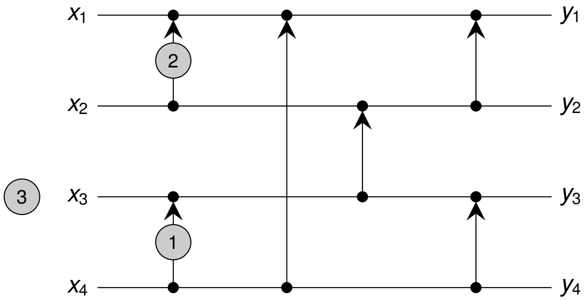

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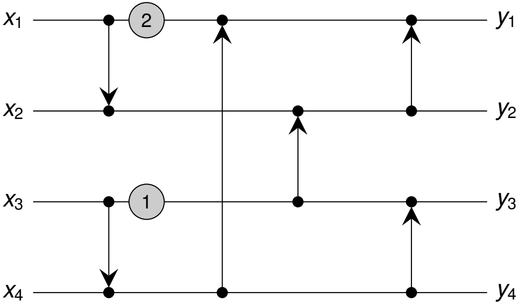

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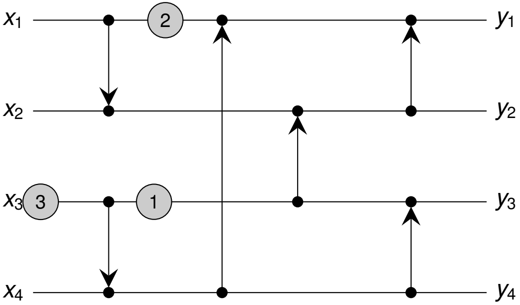

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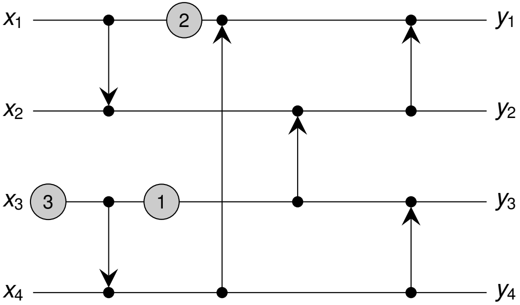

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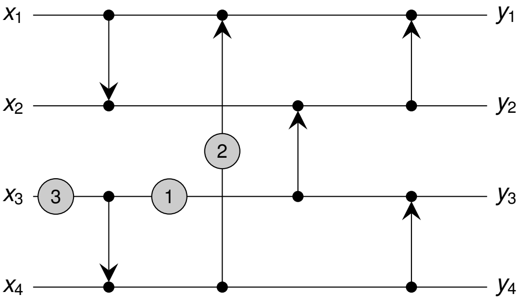

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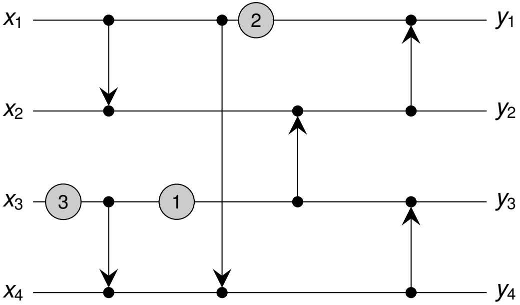

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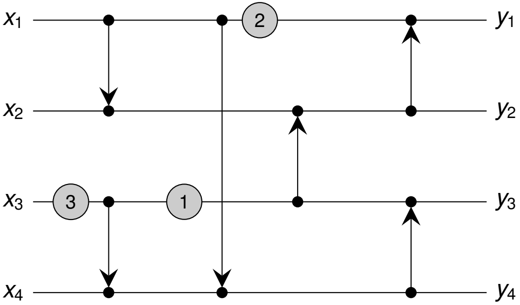

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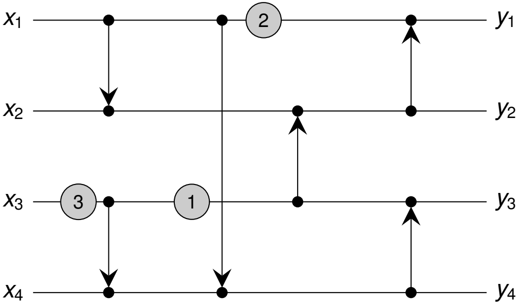

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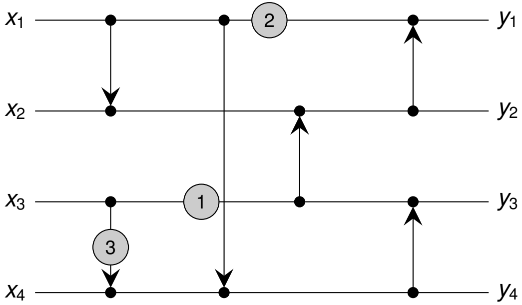

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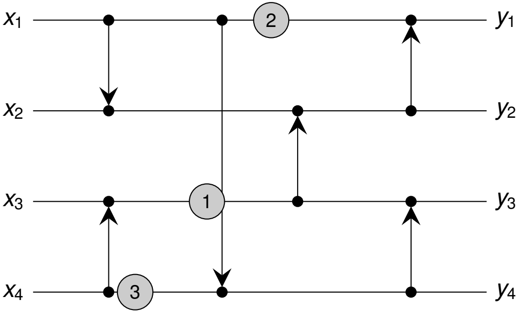

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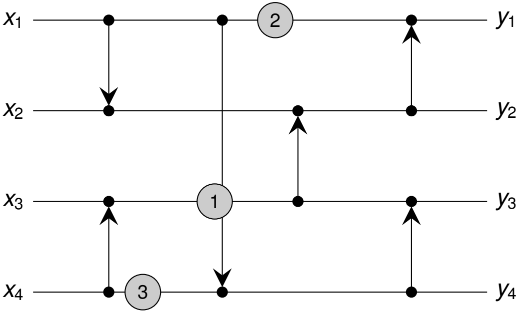

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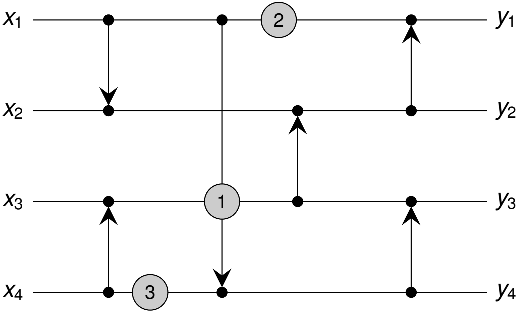

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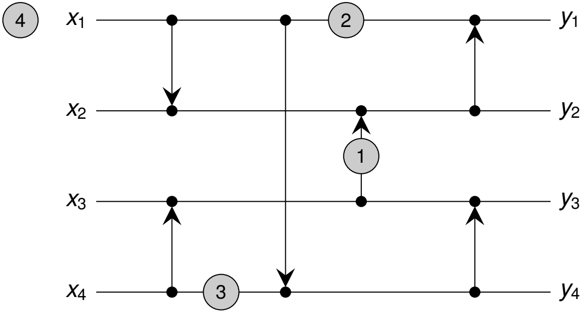

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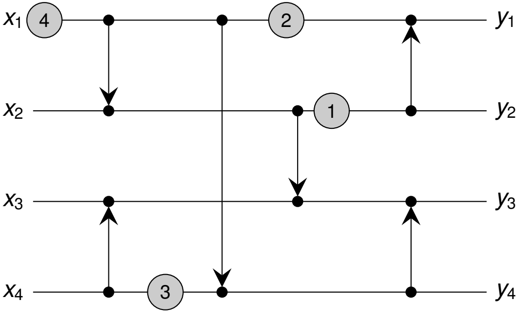

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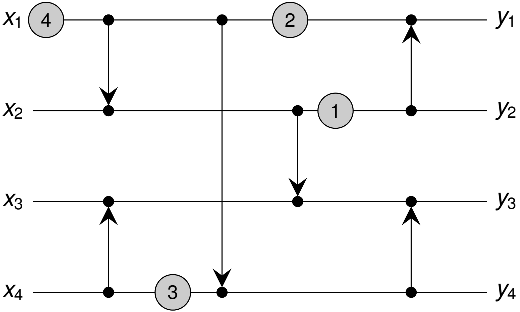

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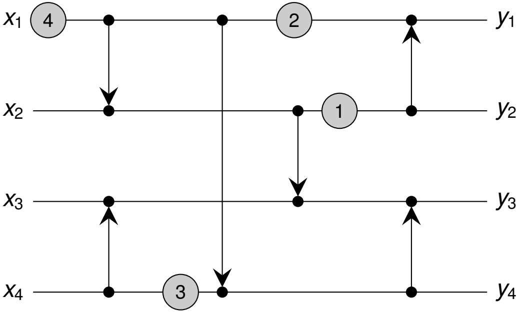

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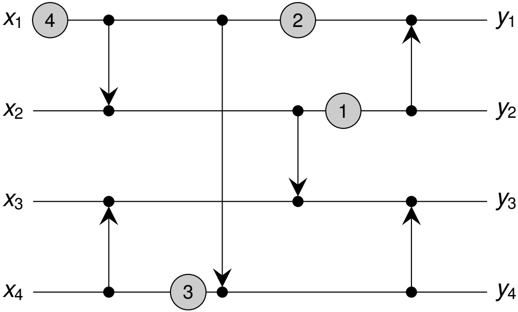

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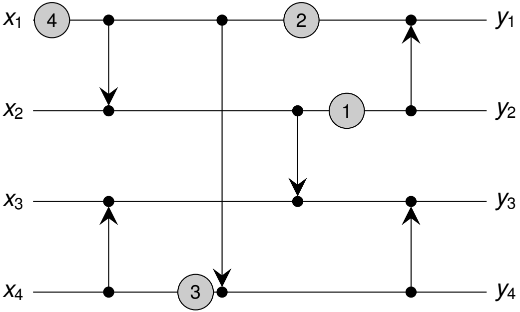

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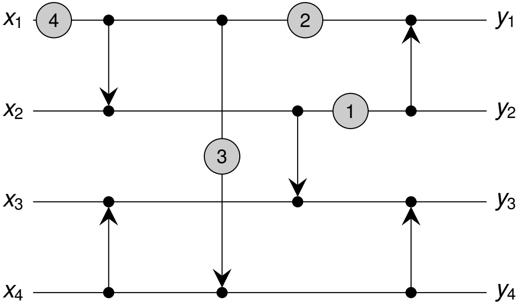

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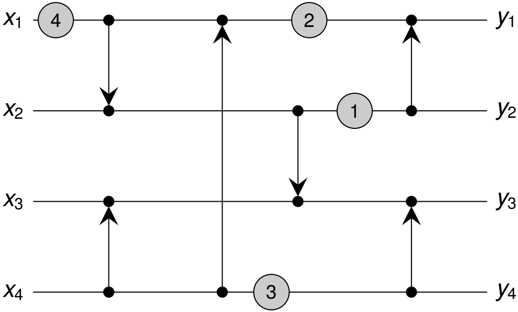

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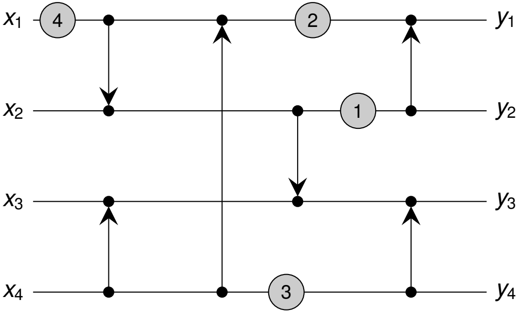

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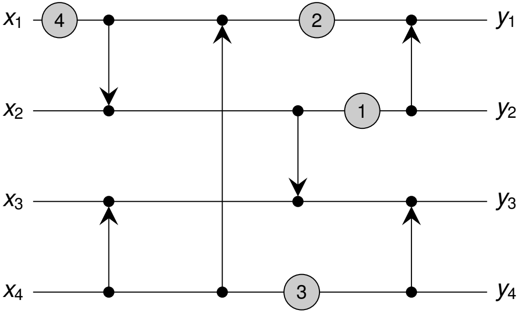

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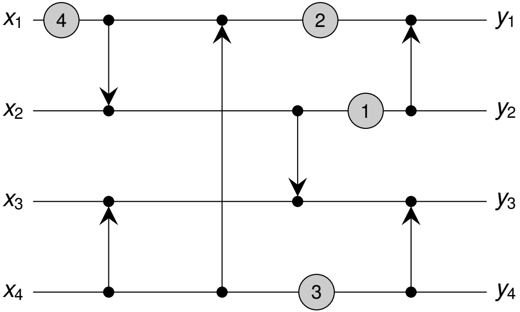

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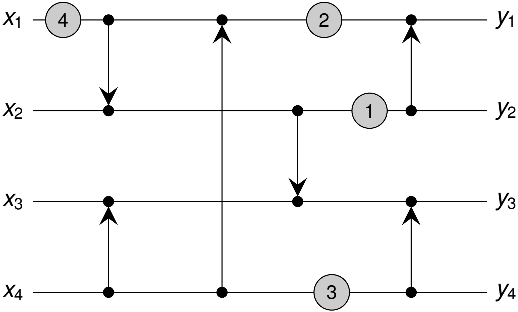

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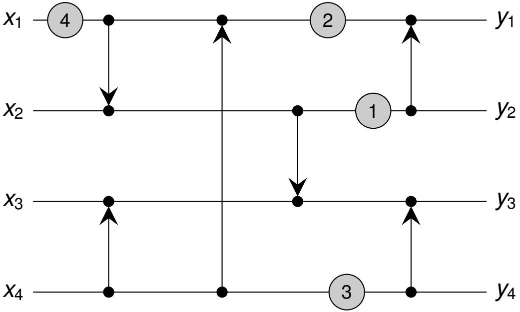

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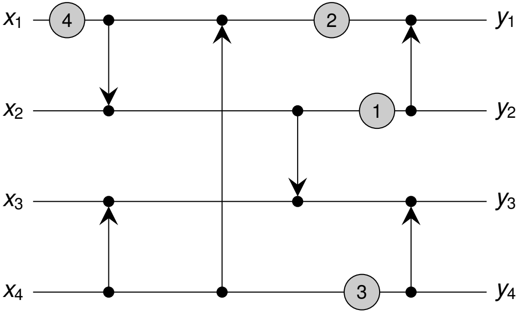

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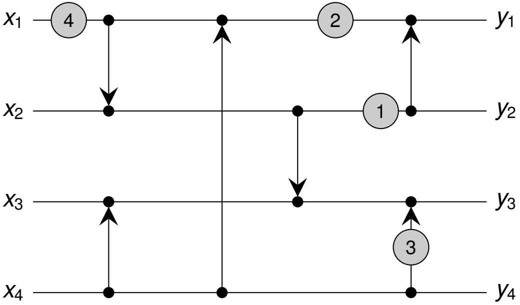

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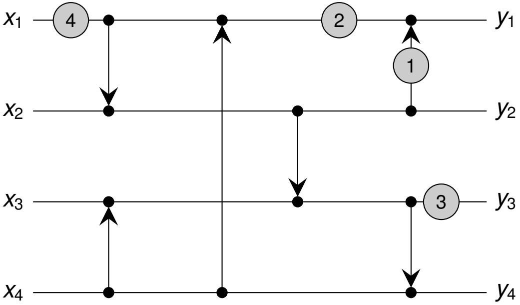

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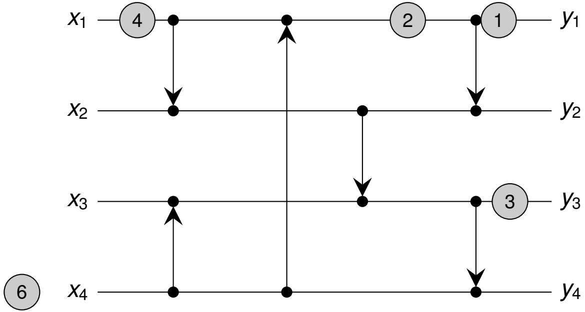

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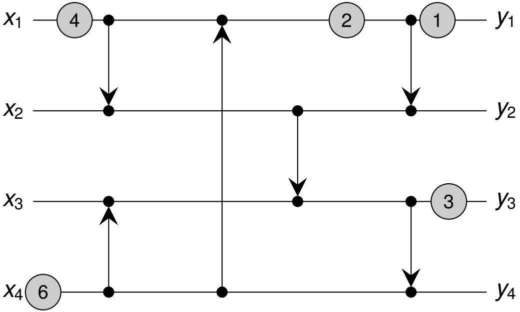

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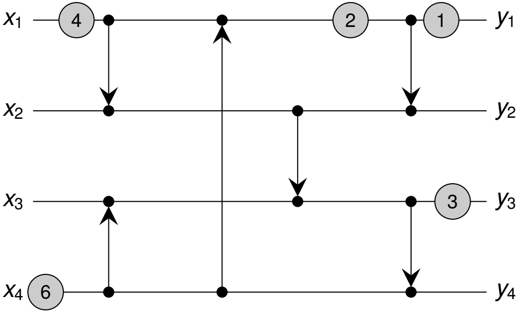

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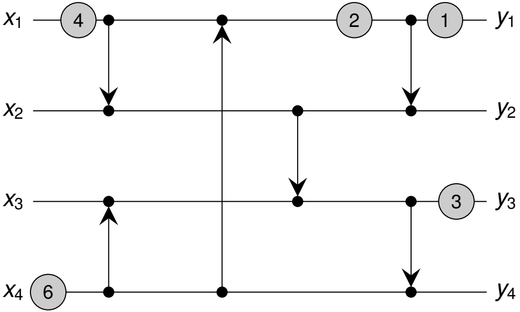

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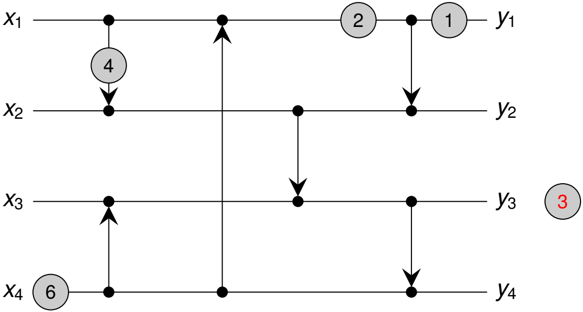

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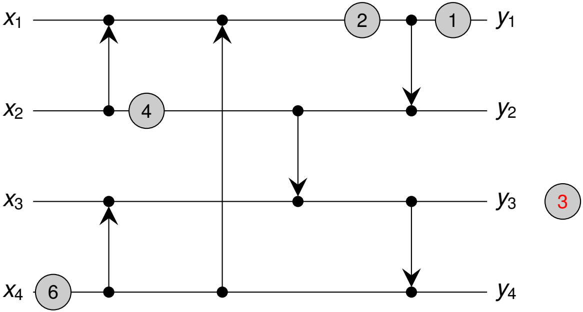

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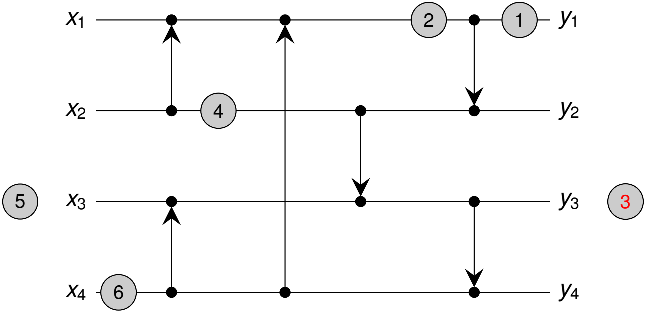

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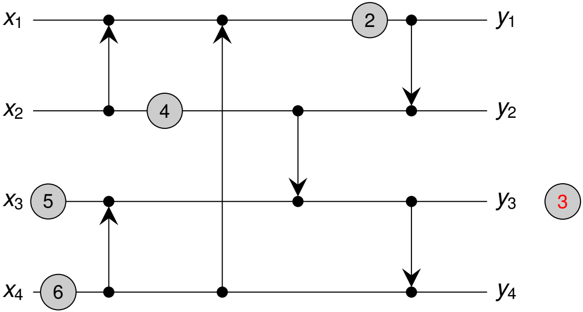

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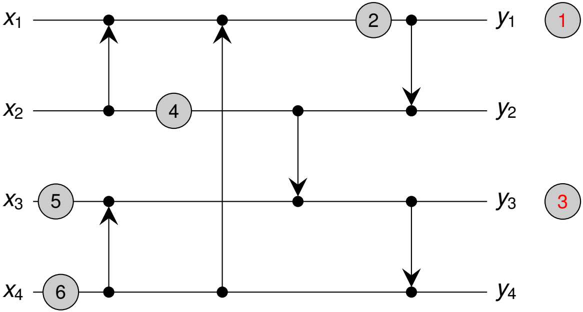

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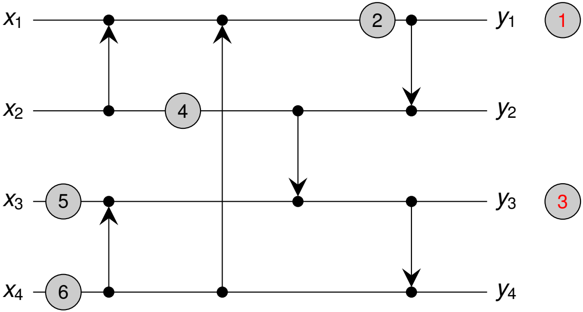

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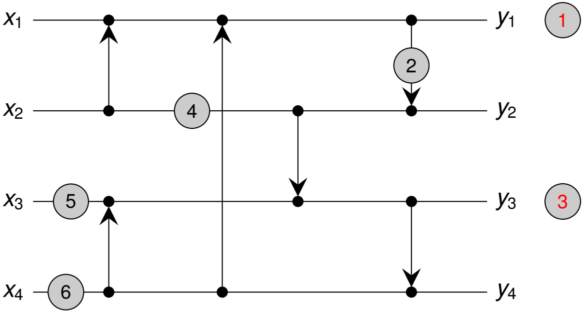

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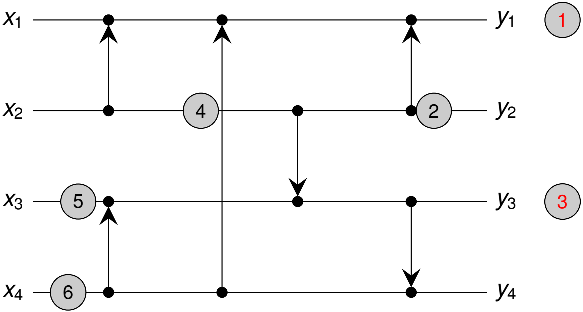

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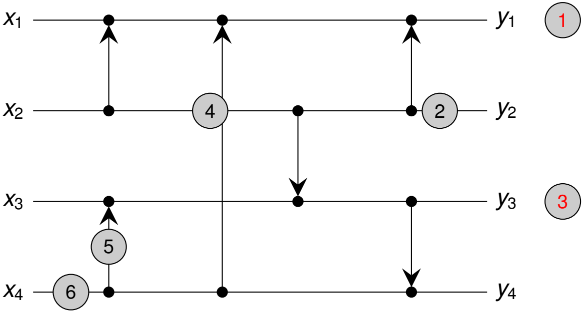

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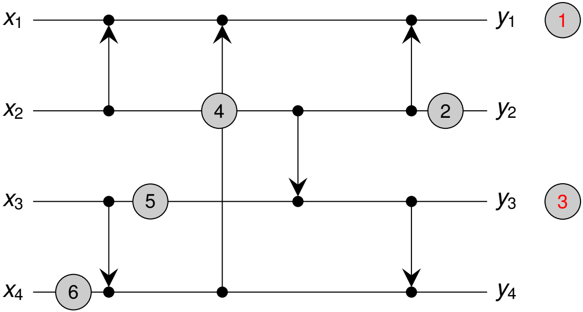

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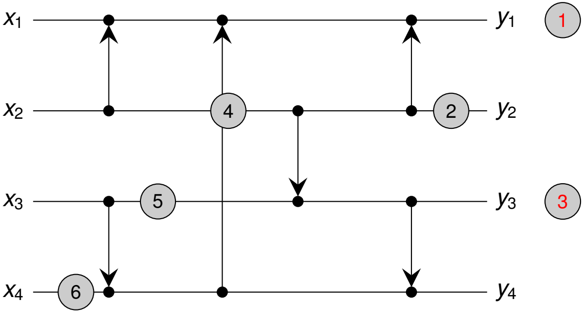

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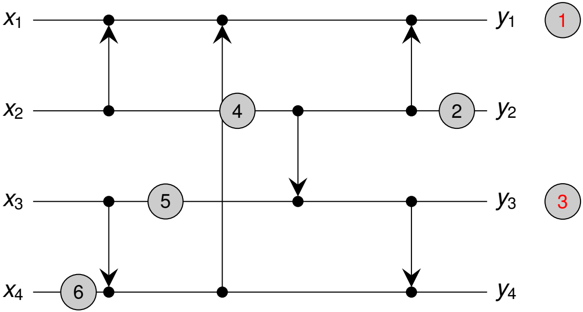

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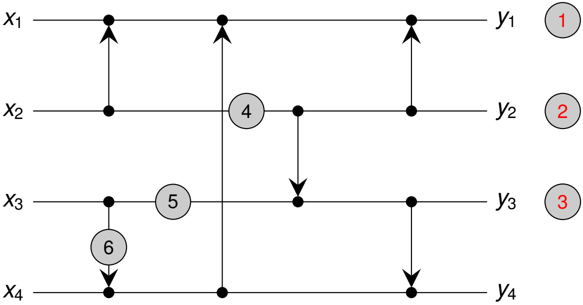

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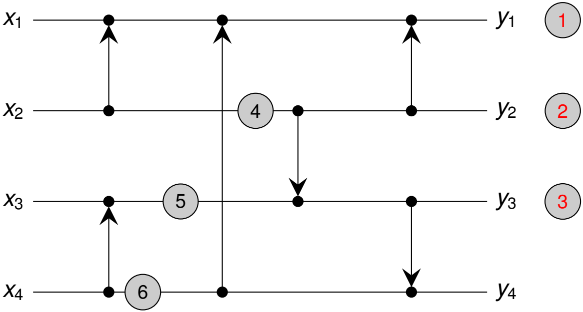

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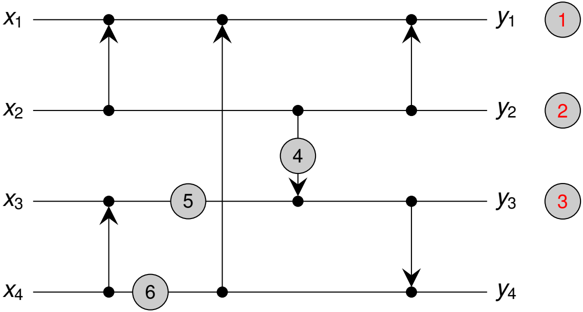

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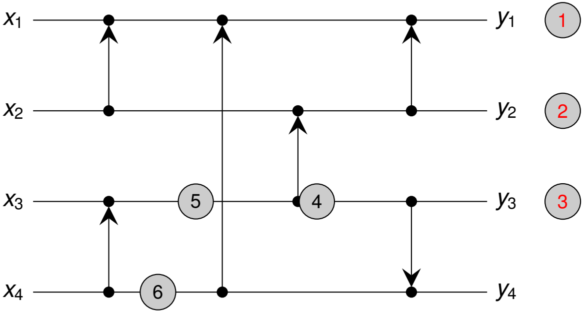

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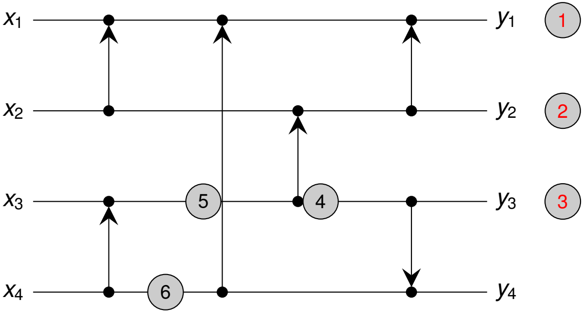

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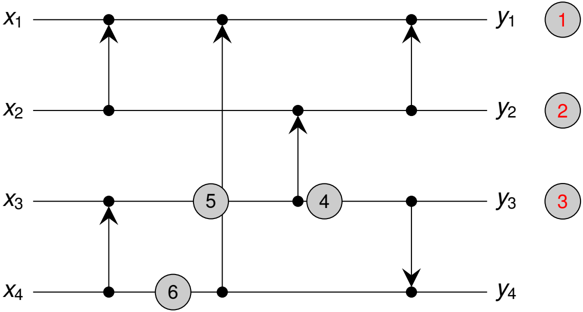

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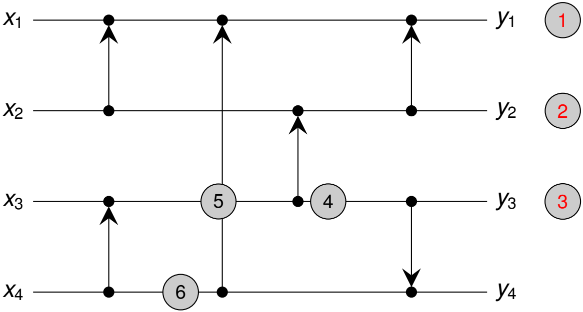

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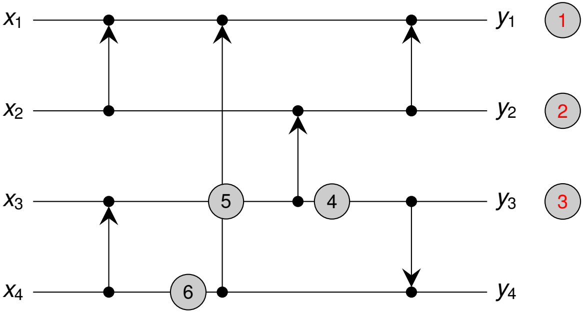

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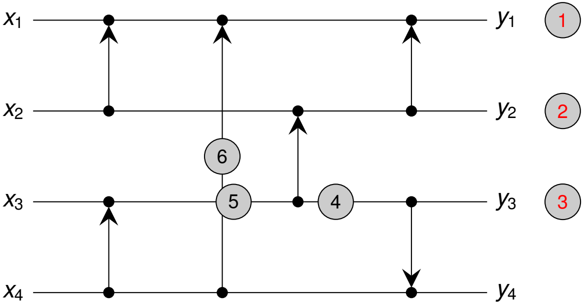

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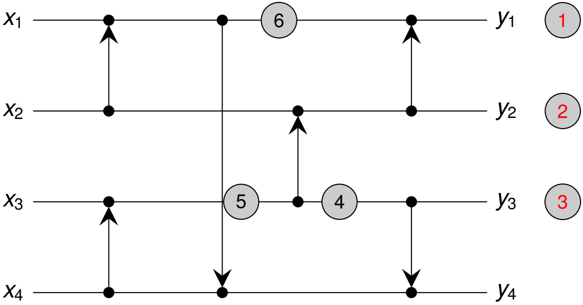

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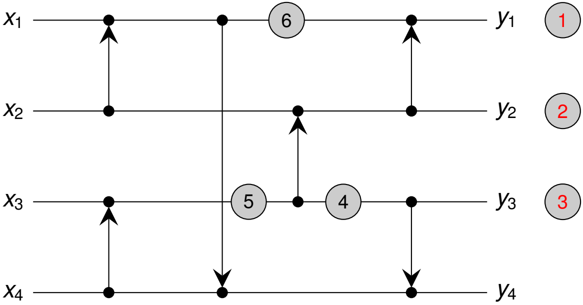

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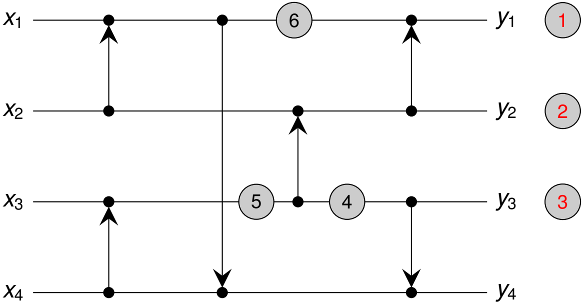

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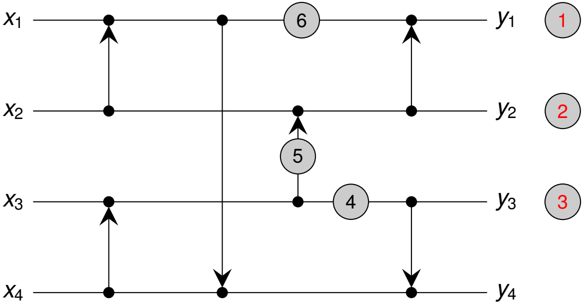

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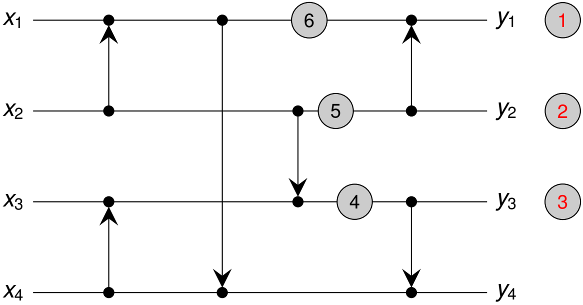

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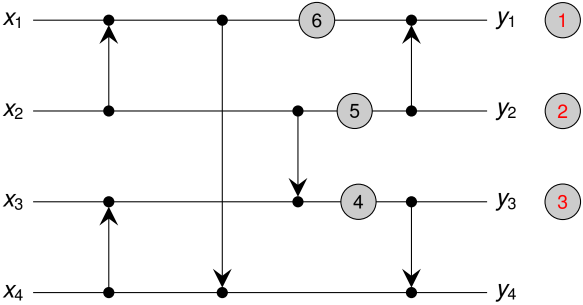

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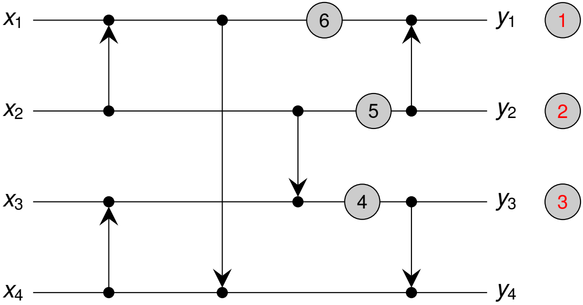

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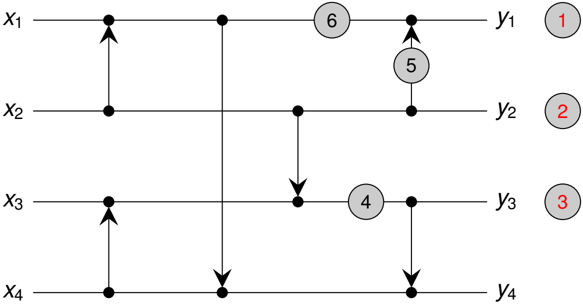

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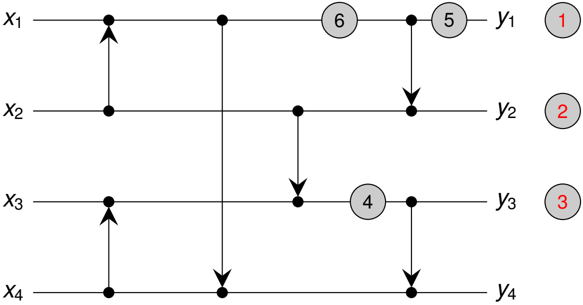

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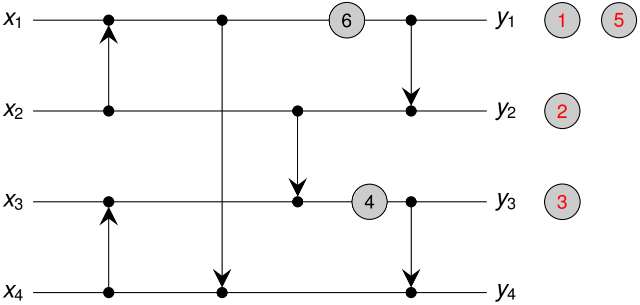

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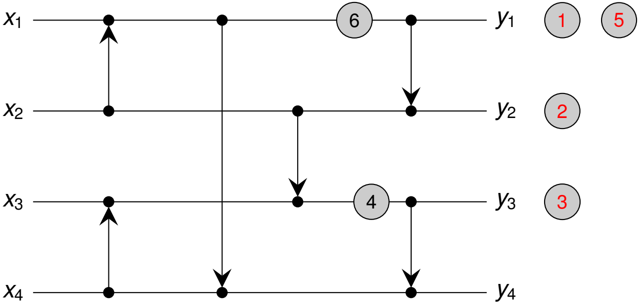

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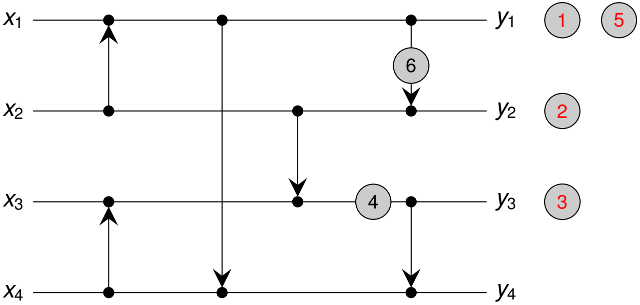

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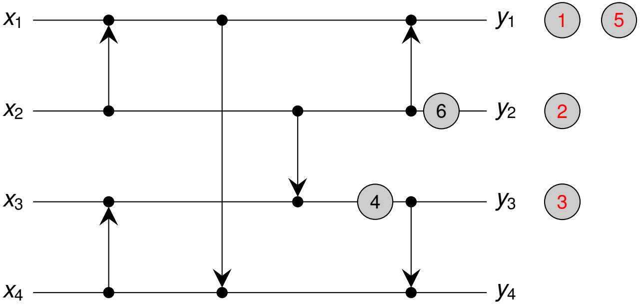

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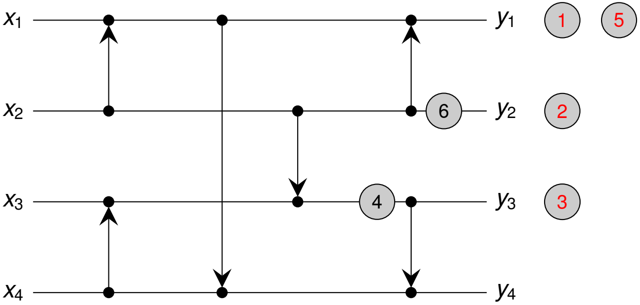

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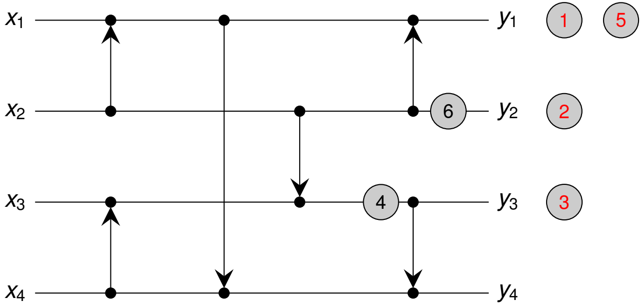

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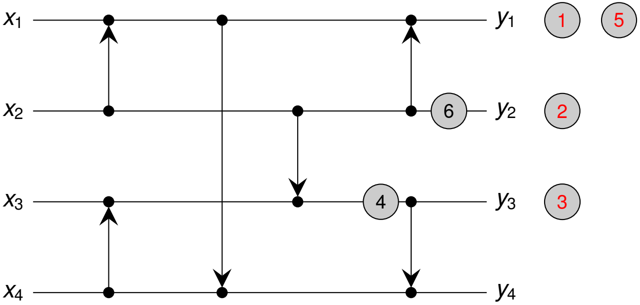

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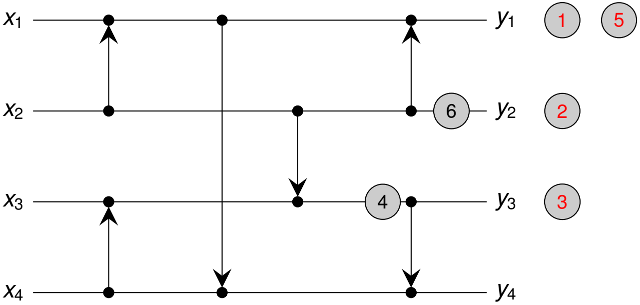

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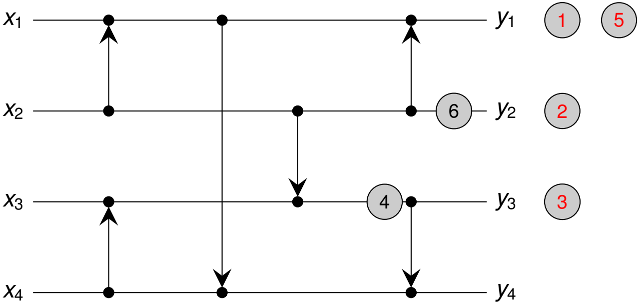

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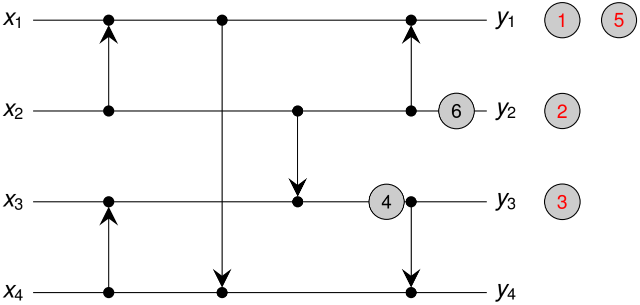

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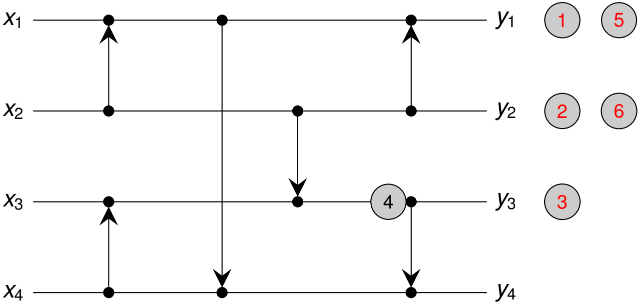

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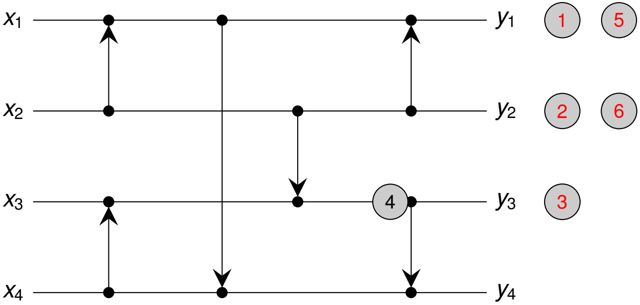

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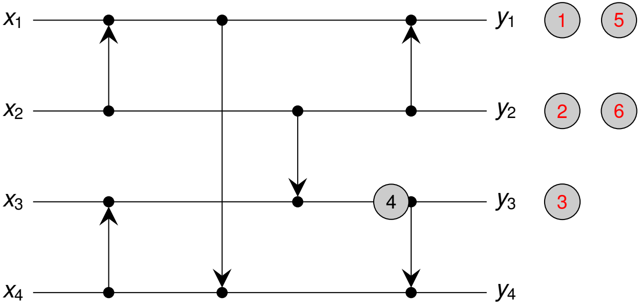

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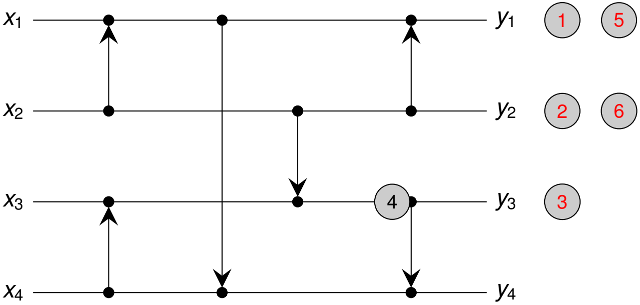

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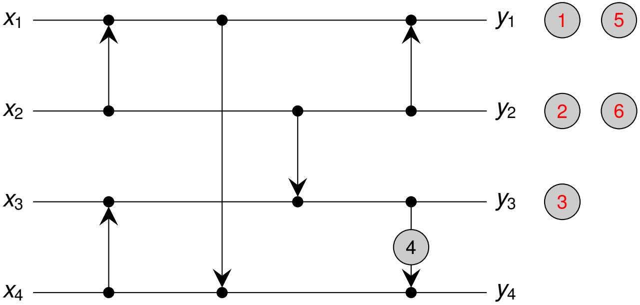

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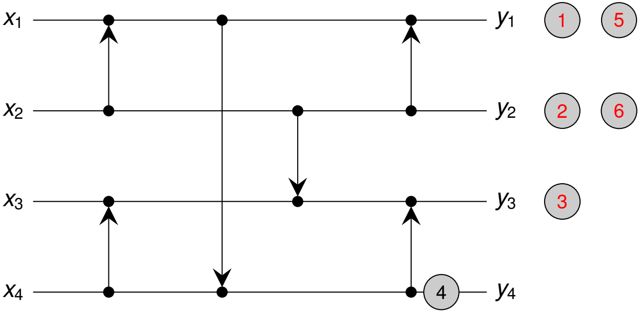

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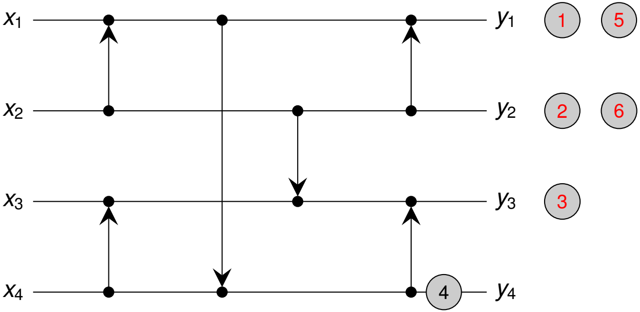

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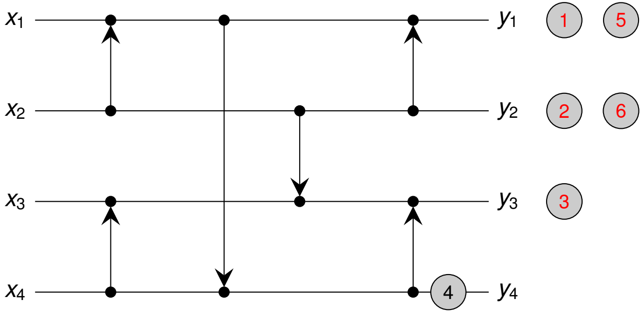

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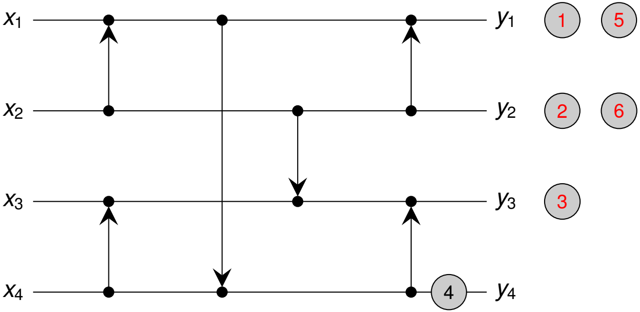

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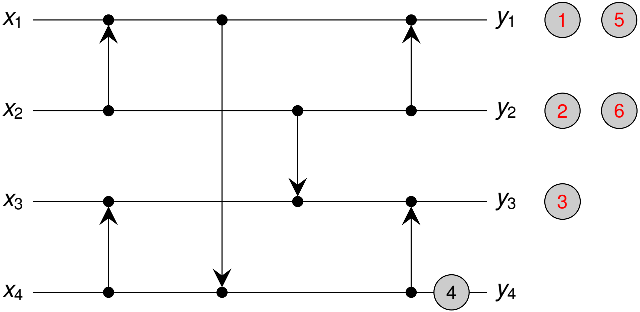

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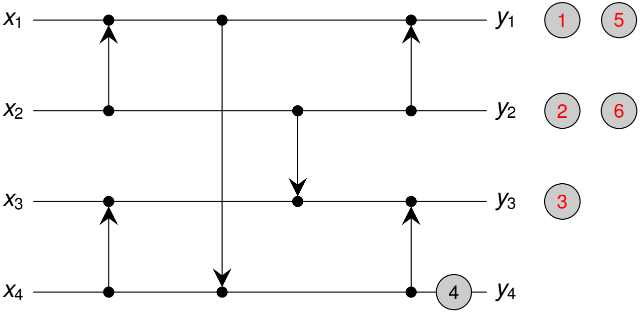

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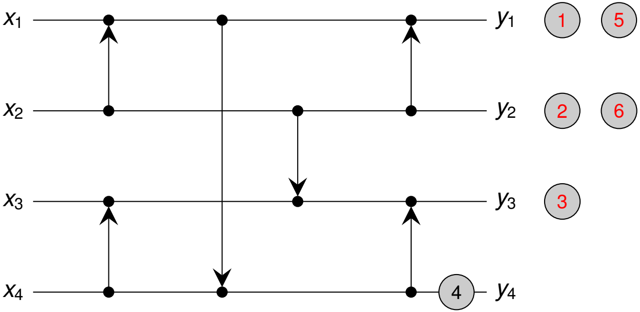

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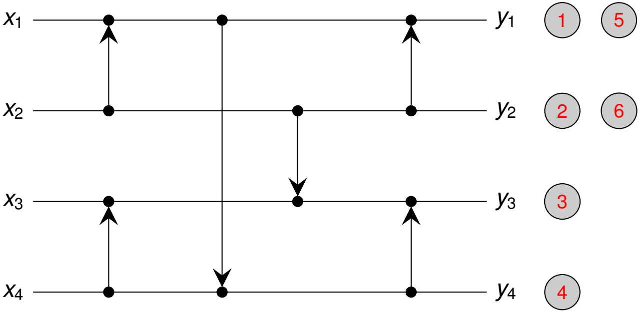

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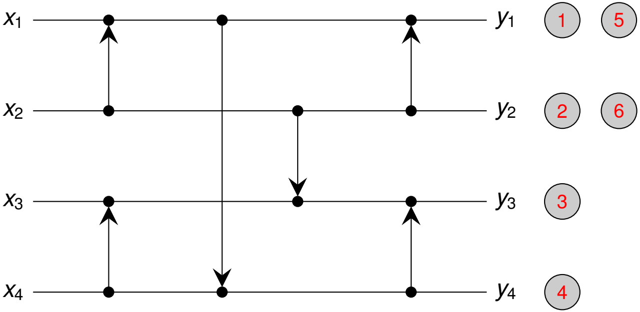
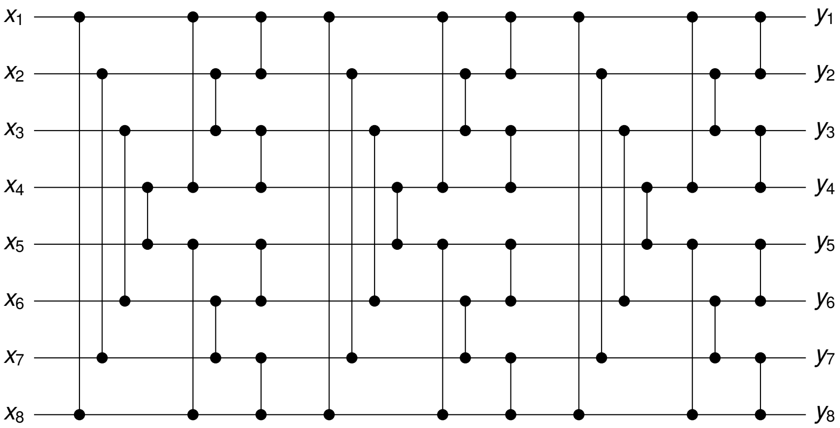


Counting can be done as follows:
Add **local counter** to each output wire i, to assign consecutive numbers $i, i + n, i + 2 \cdot n, \ldots$

A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



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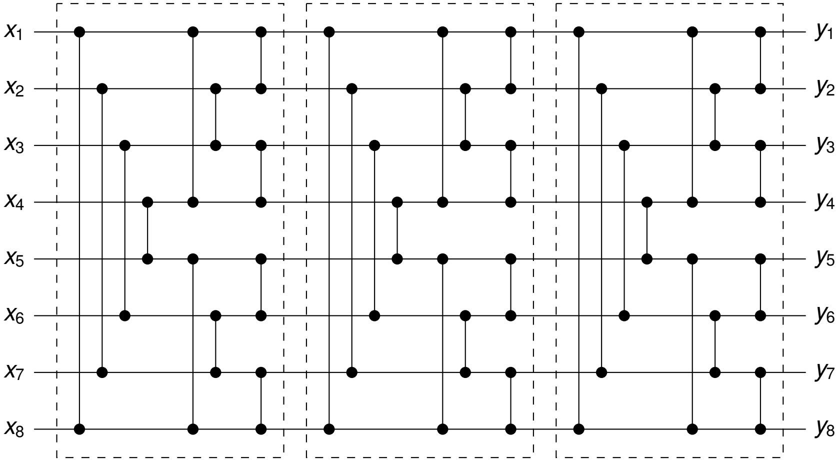

Consists of $\log n$ BLOCK$[n]$ networks each of which has depth $\log n$

From Counting to Sorting

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

From Counting to Sorting

The converse is not true!

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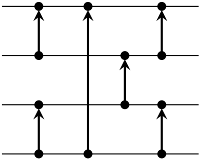

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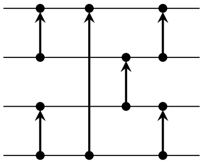

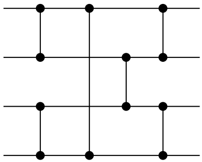

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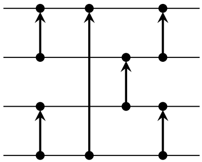

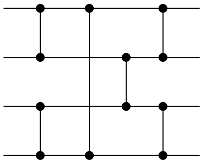

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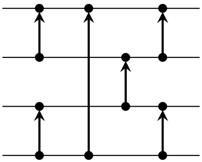

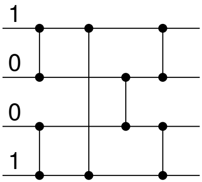

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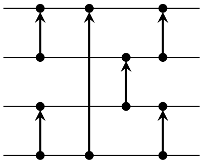

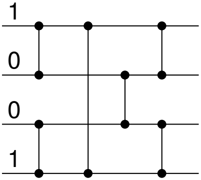

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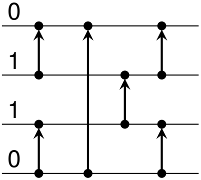

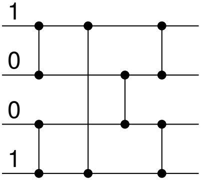

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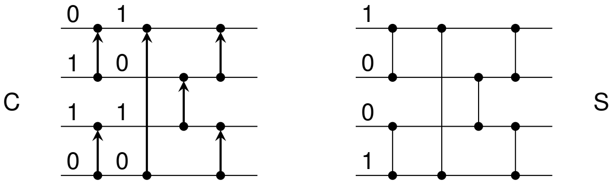

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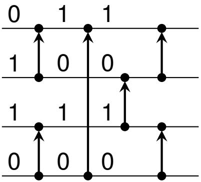

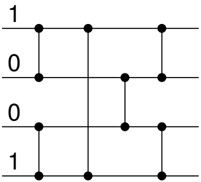

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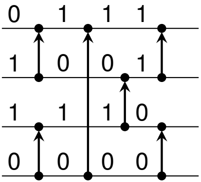

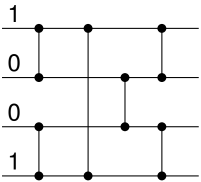

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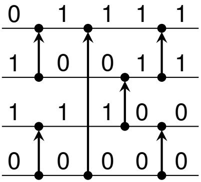

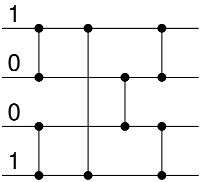

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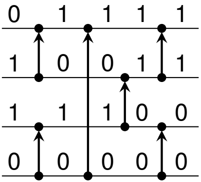

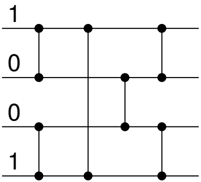

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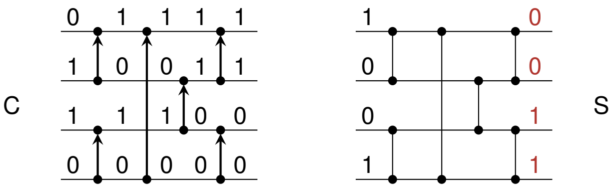

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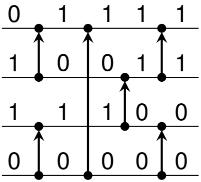
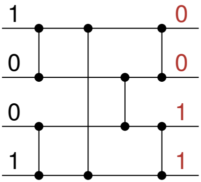
- By the Zero-One Principle, S is a sorting network. $\square$

C





S

II. Matrix Multiplication

Thomas Sauerwald

Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Serial Matrix Multiplication

Digression: Multithreading

Multithreaded Matrix Multiplication

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \dots, n.$$

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```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
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SQUARE-MATRIX-MULTIPLY(A, B) takes time $\Theta(n^3)$.

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This definition suggests that $n^2 \cdot n = n^3$ arithmetic operations are necessary.

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Divide & Conquer: First Approach

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This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

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$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Each equation specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their products.

Divide & Conquer: First Approach (Pseudocode)

$$
\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
C_{11} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}
\end{aligned}
$$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
$$C_{11} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

1 $n = A.rows$
2 let C be a new $n \times n$ matrix
3 **if** $n == 1$
4 $\quad c_{11} = a_{11} \cdot b_{11}$
5 **else** partition A, B, and C as in equations (4.9)
6 $\quad C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
7 $\quad C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
8 $\quad C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
9 $\quad C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
10 **return** C

Line 5: Handle submatrices implicitly through index calculations instead of creating them.

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
$$C_{11} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure.

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ & \text{if } n > 1. \end{cases}$$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ & \text{if } n > 1. \end{cases}$$

8 Multiplications

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c_11 = a_11 · b_11
else partition A, B, and C as in equations (4.9)
    C_11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_11, B_11)
           + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_12, B_21)
    C_12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_11, B_12)
           + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_12, B_22)
    C_21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_21, B_11)
           + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_22, B_21)
    C_22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_21, B_12)
           + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_22, B_22)
return C
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

4 Additions and Partitioning

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

4 Additions and Partitioning

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) =$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n})$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

No improvement over the naive algorithm!

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

```
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
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          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

Goal: Reduce the number of multiplications

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Strassen's Algorithm (1969)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices
2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.
3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$
4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Strassen's Algorithm (1969)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices
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3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$
4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

Time for steps 1,2,4: $\Theta(n^2)$, hence $T(n) = 7 \cdot T(n/2) + \Theta(n^2) \Rightarrow T(n) = \Theta(n^{\log 7})$.

Solving the Recursion

$T(n) = 7 \cdot T(n/2) + c \cdot n^2$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
P_2 &= S_2 \cdot B_{22} = (A_{11} + A_{12}) \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} = (A_{21} + A_{22}) \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}
$$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

$$\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
P_2 &= S_2 \cdot B_{22} = (A_{11} + A_{12}) \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} = (A_{21} + A_{22}) \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}$$

Claim

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
P_2 &= S_2 \cdot B_{22} = (A_{11} + A_{12}) \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} = (A_{21} + A_{22}) \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}
$$

Claim

$$
\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}
$$

Proof:

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

$$\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
P_2 &= S_2 \cdot B_{22} = (A_{11} + A_{12}) \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} = (A_{21} + A_{22}) \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}$$

Claim

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Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

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Current State-of-the-Art

Open Problem: Is there an algorithm with quadratic complexity?

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Outline

Memory Models

Distributed Memory

- Each processor has its private memory
- Access to memory of another processor via messages

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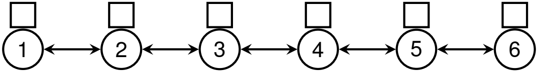

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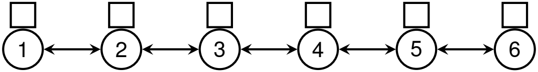

Shared Memory

- Central location of memory
- Each processor has direct access

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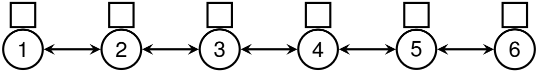

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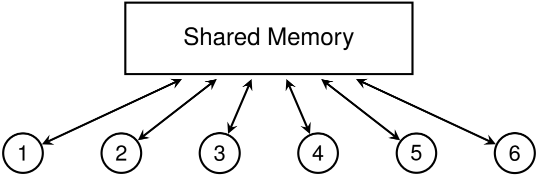

Dynamic Multithreading

- Programming shared-memory parallel computer difficult

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- Use concurrency platform which coordinates all resources

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Scheduling jobs, communication protocols, load balancing etc.

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Functionalities:

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- `spawn`

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- **`spawn`**
  - (optional) prefix to a procedure call statement
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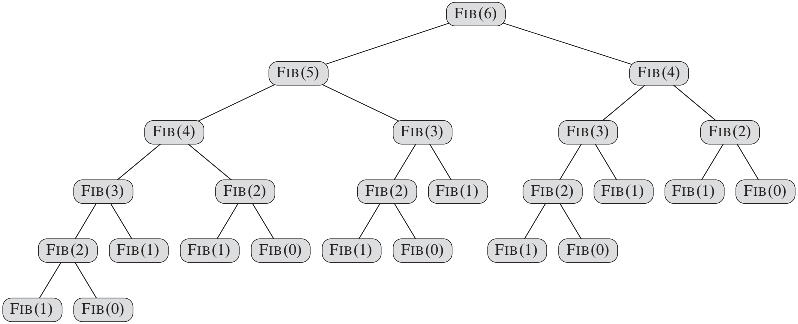
Only logical parallelism, but not actual!
Need a scheduler to map threads to processors.

Computing Fibonacci Numbers Recursively (Fig. 27.1)

```
FIB(n)
  if n<=1 return n
  else x=FIB(n-1)
       y=FIB(n-2)
       return x+y
```

Computing Fibonacci Numbers Recursively (Fig. 27.1)

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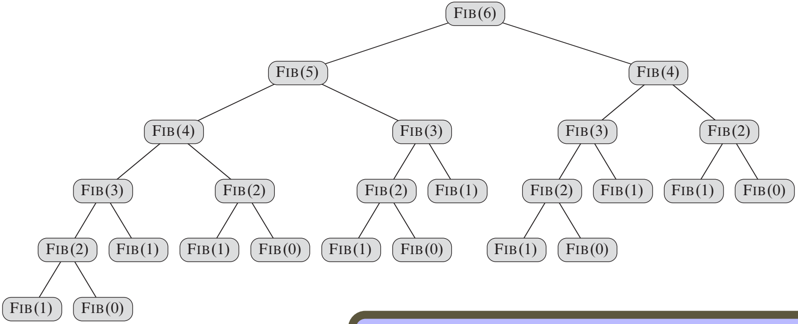

Very inefficient – exponential time!

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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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P-FIB(n)
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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

- Without **spawn** and **sync** same pseudocode as before
- **spawn** does not imply parallel execution (depends on scheduler)

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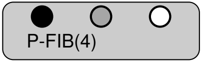


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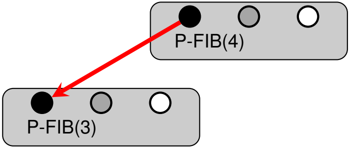


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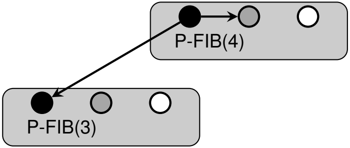


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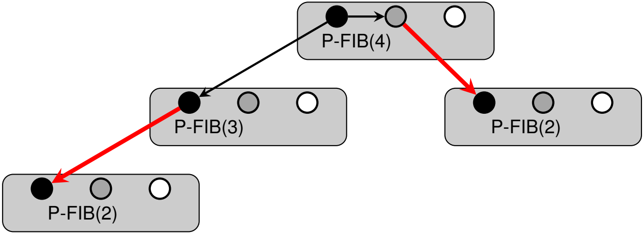


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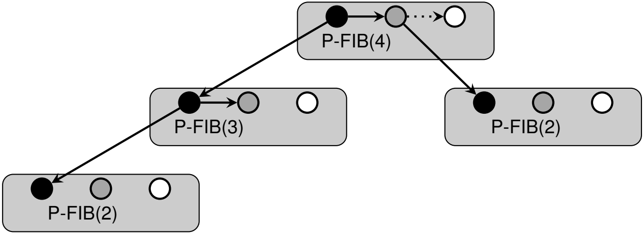


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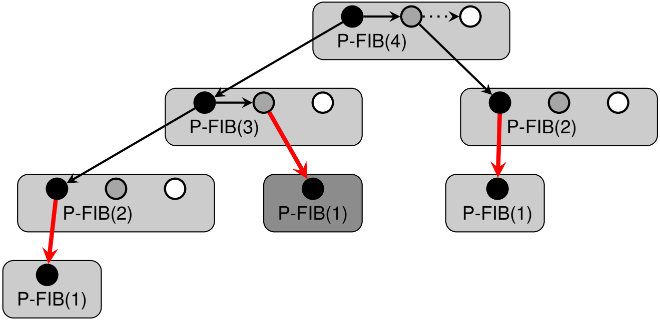


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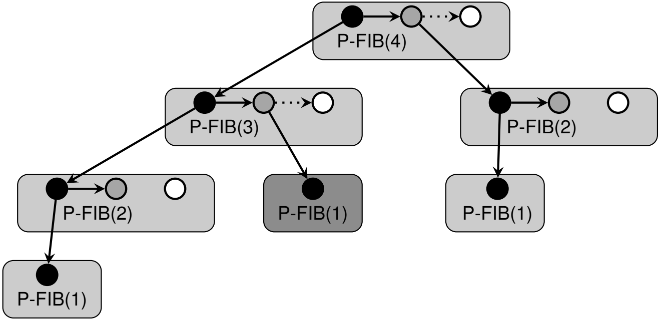


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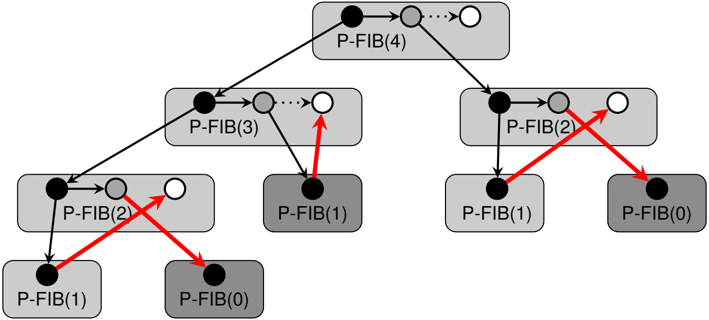


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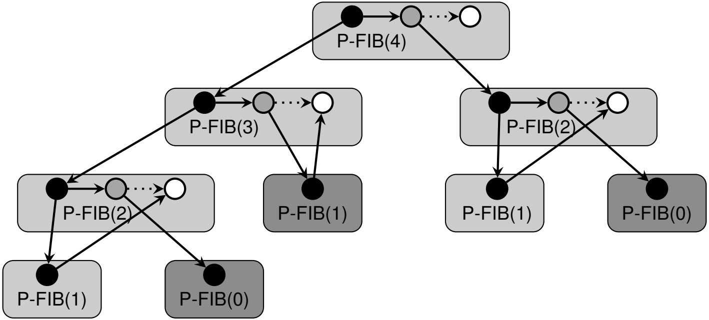


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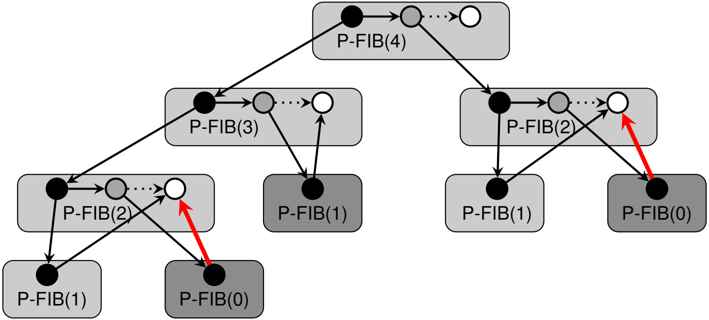


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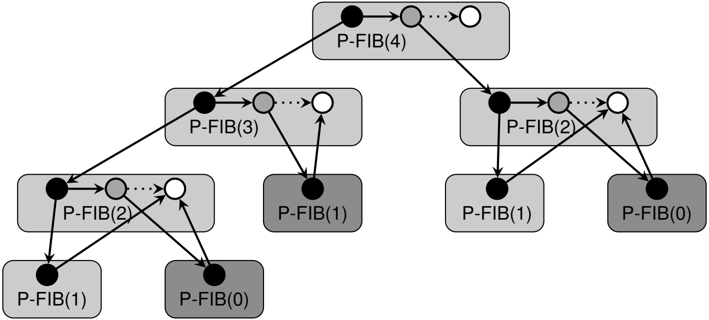
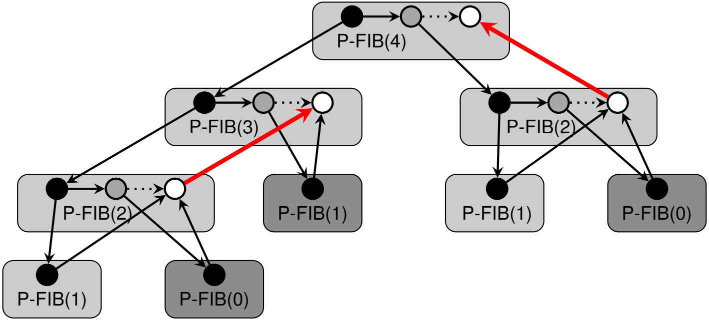



```
P-FIB(n)
   if n<=1 return n
   else x=spawn P-FIB(n-1)
        y=P-FIB(n-2)
        sync
        return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)




```
0: P-FIB(n)
1:   if n<=1 return n
2:   else x=spawn P-FIB(n-1)
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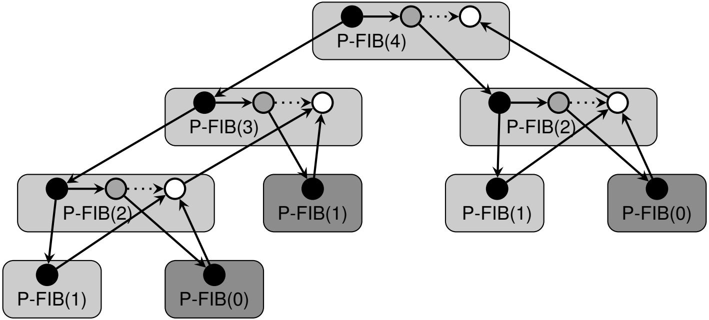
5:        return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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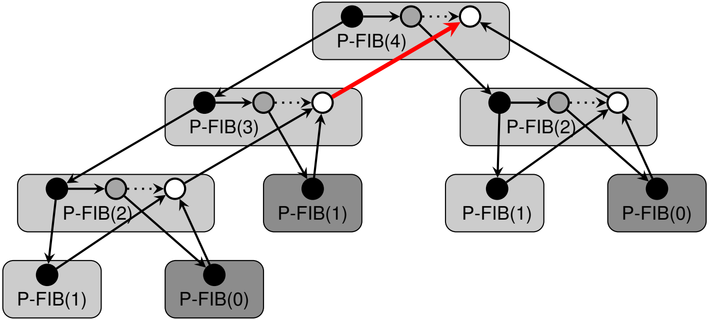


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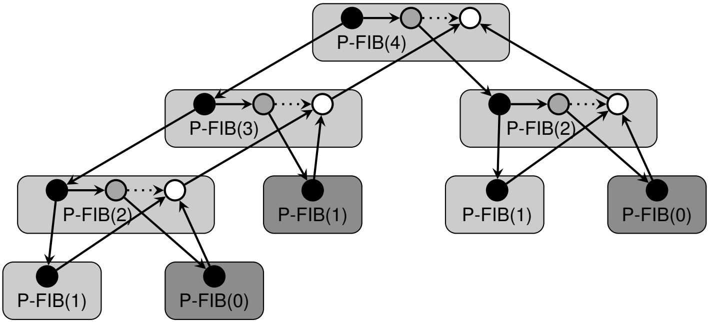


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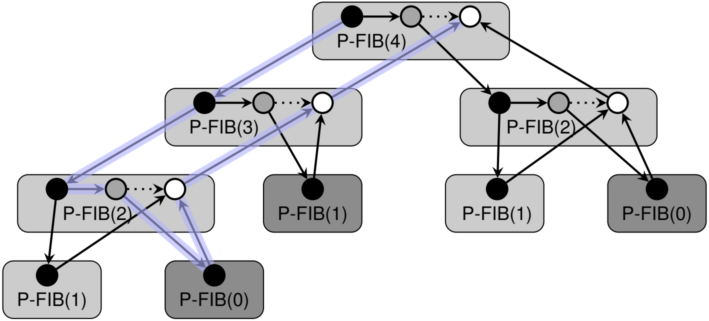


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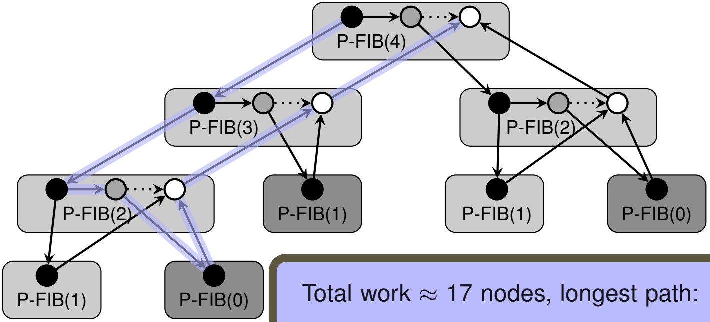

Total work $\approx$ 17 nodes, longest path: 8 nodes

```
P-FIB(n)
  if n<=1 return n
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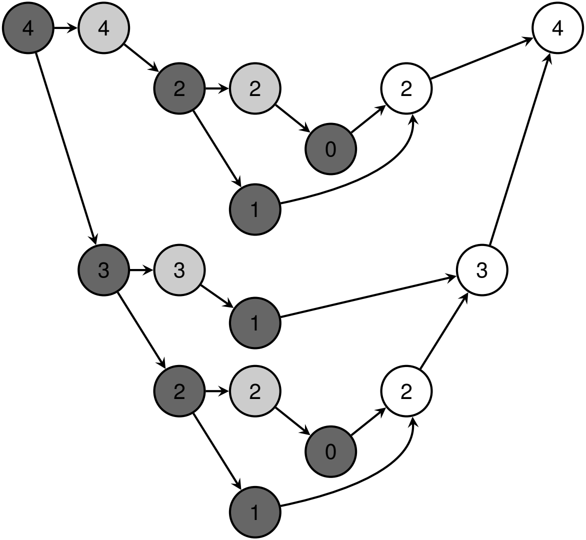
       sync
       return x+y
```

Computing Fibonacci Numbers in Parallel (DAG Perspective)



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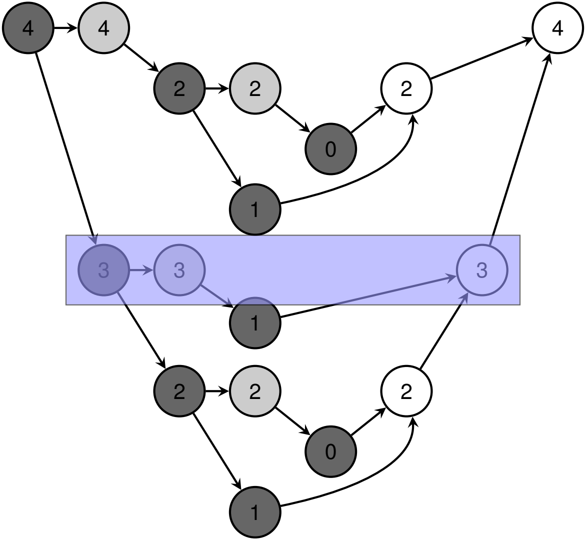

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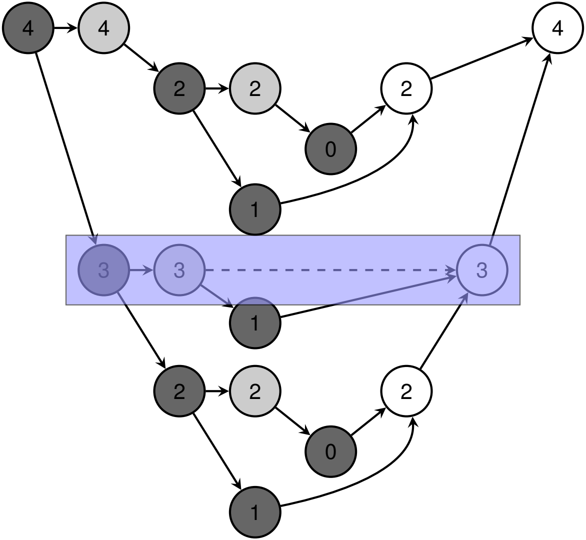

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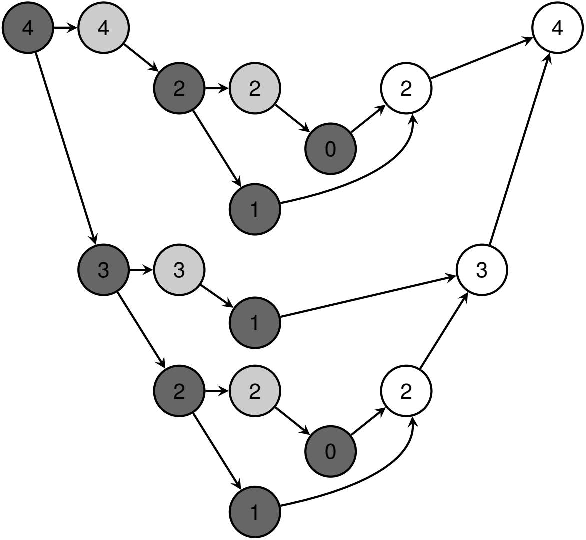

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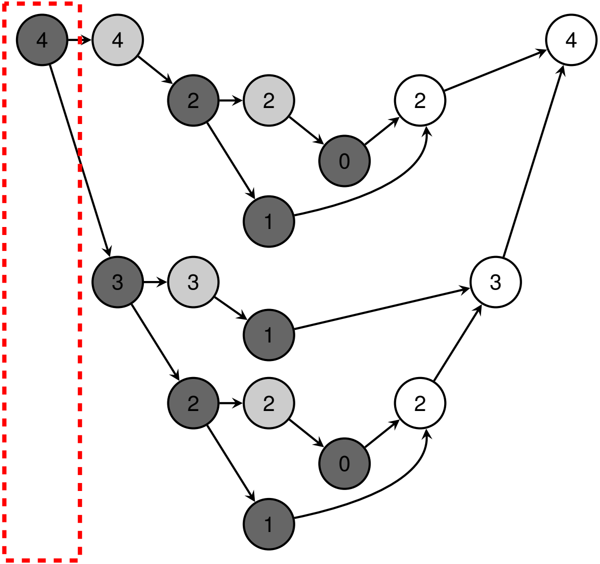

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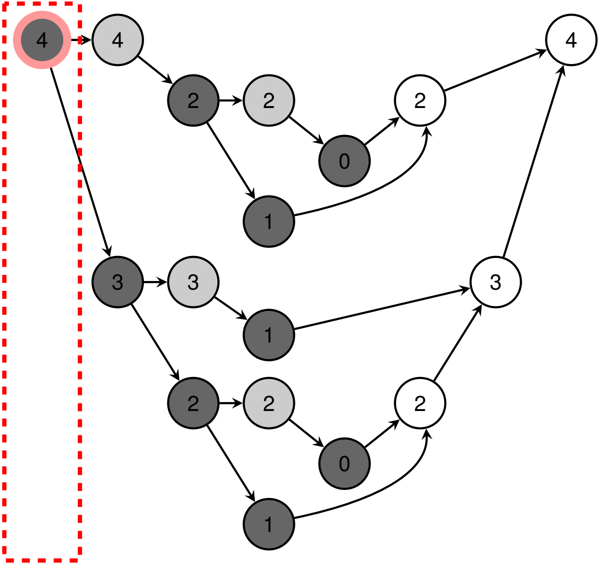

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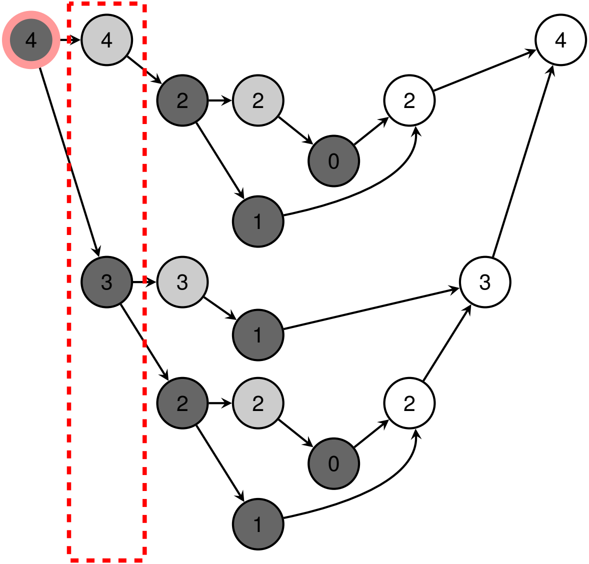

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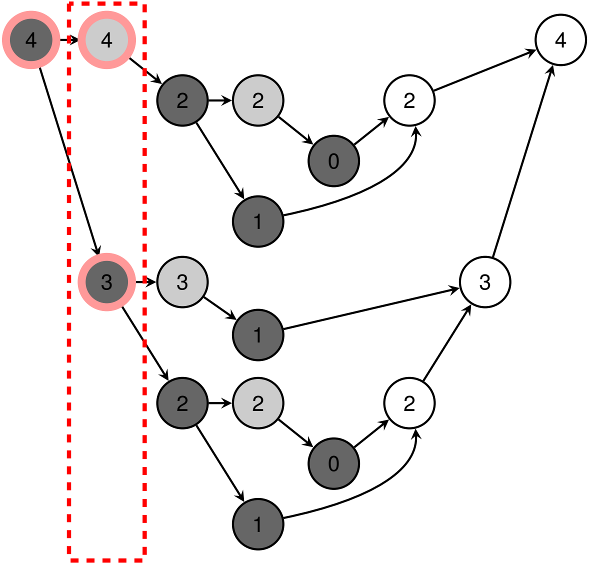

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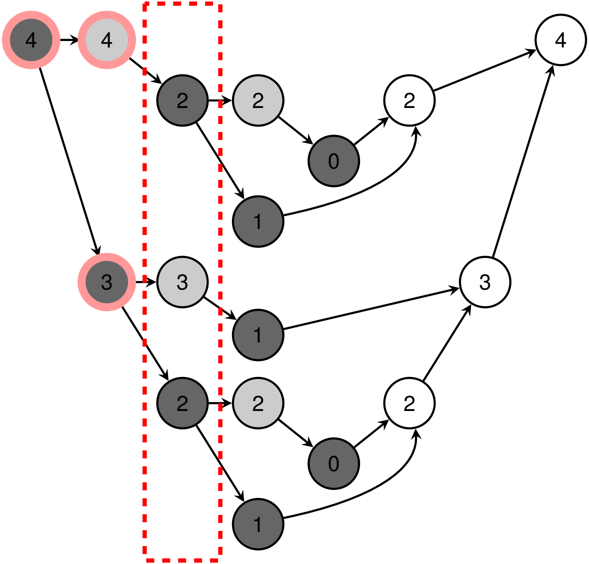

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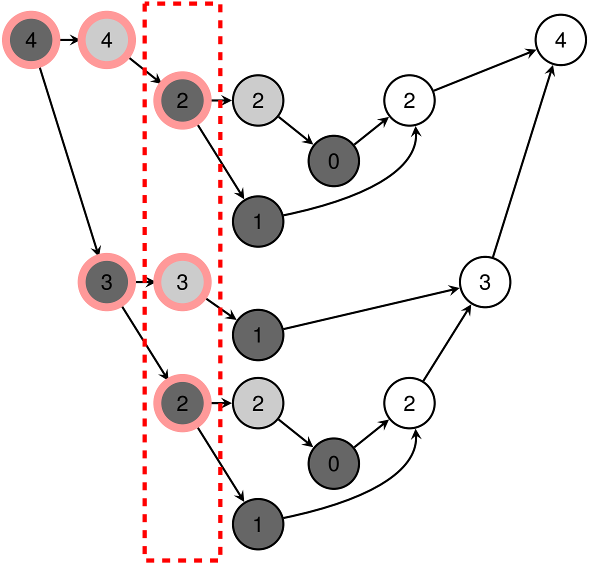

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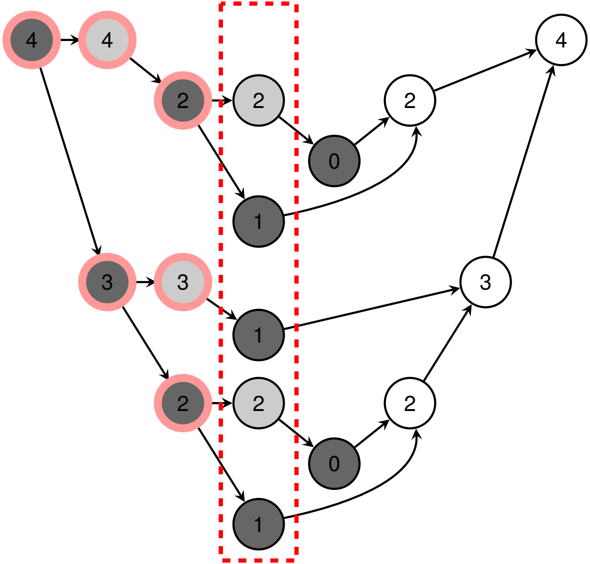

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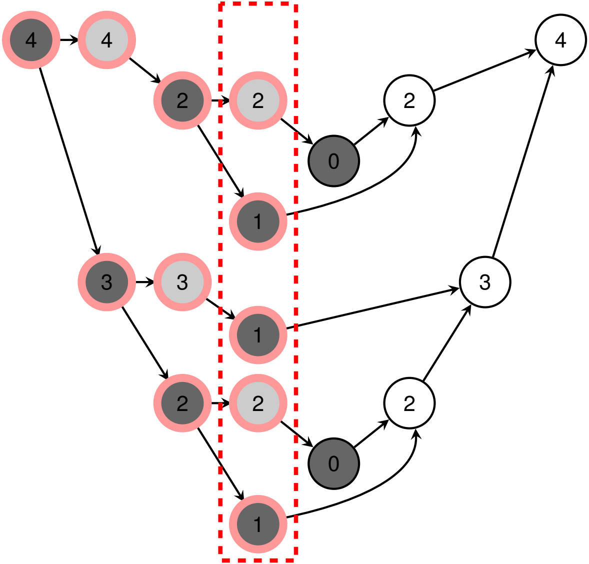

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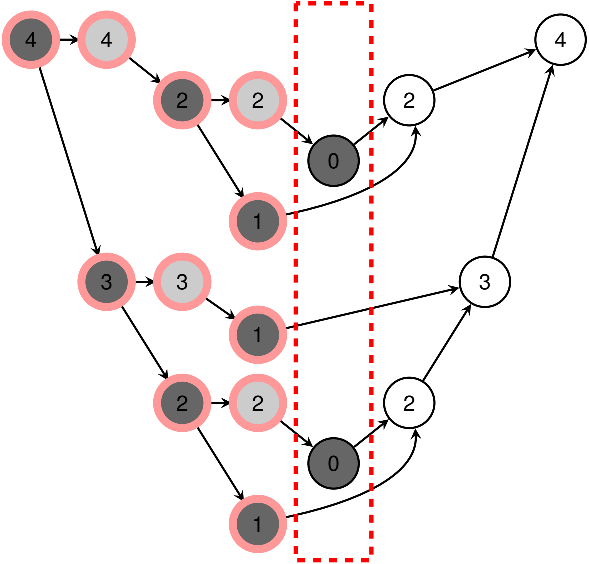

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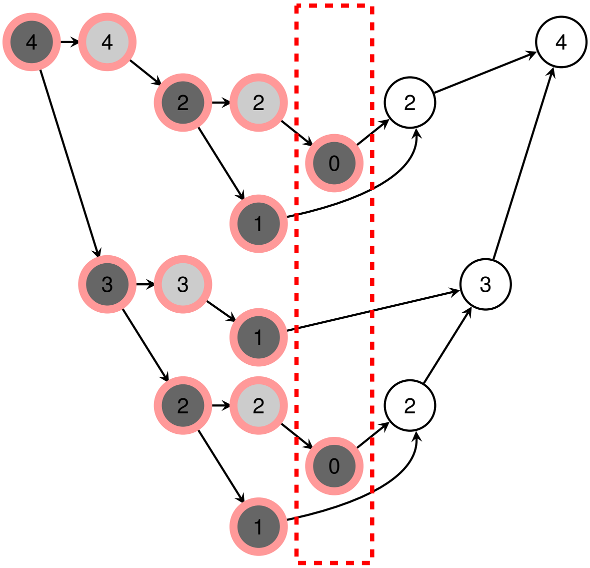

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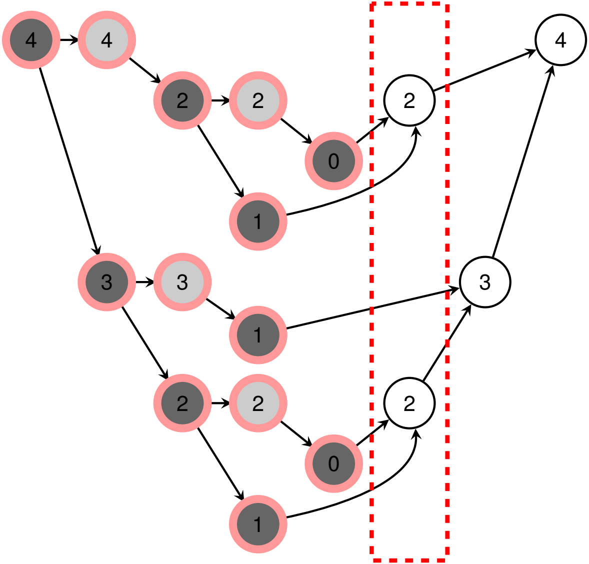

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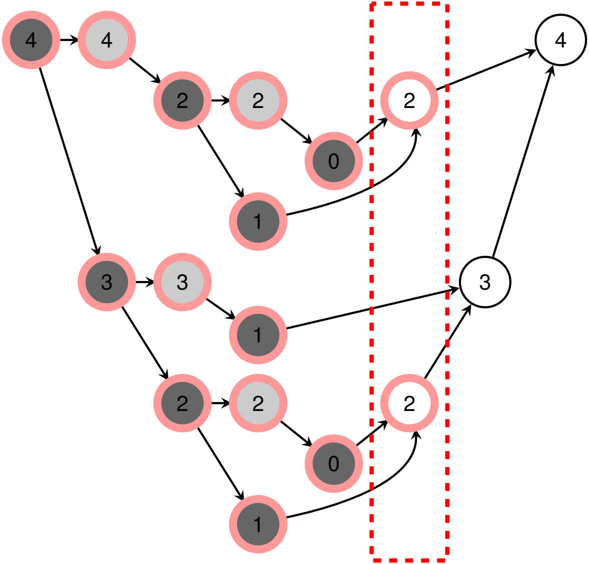

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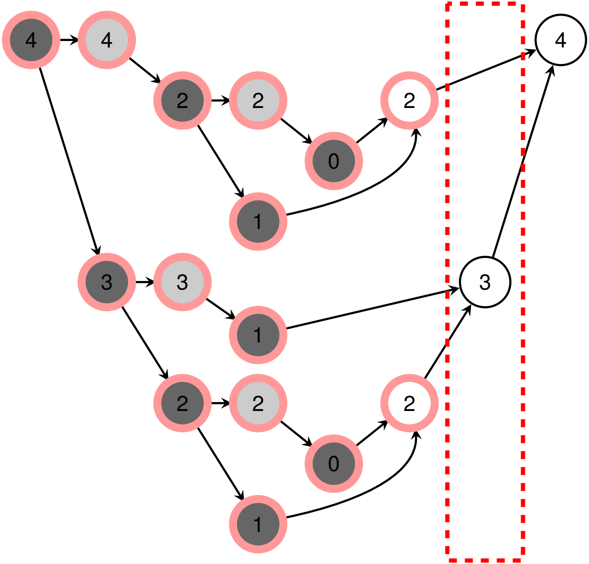

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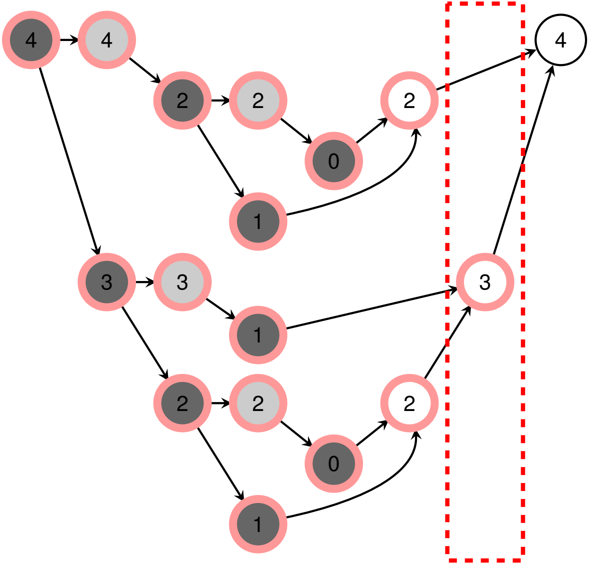

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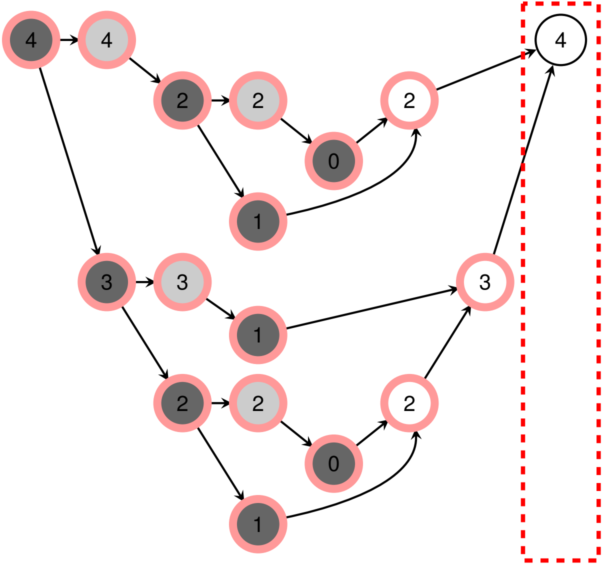

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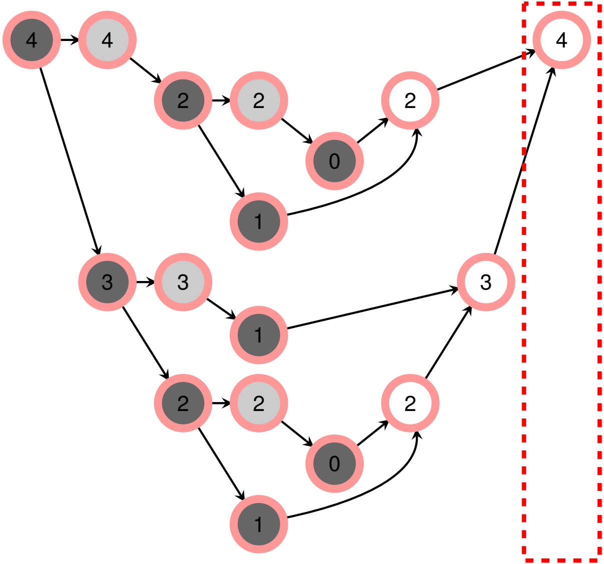

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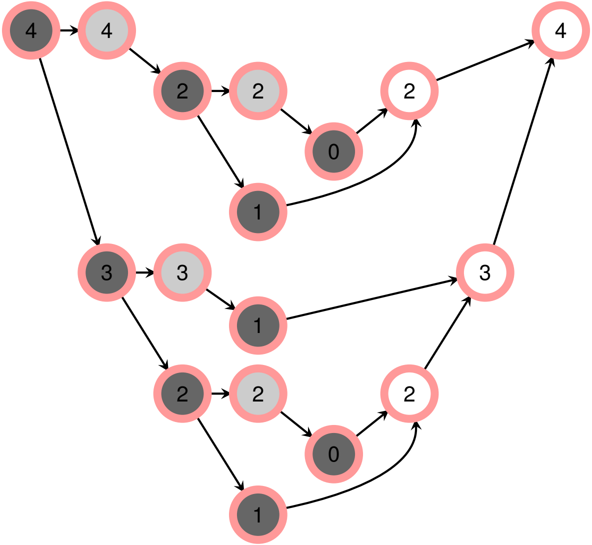

Performance Measures

Work

Total time to execute everything on a single processor.

Performance Measures

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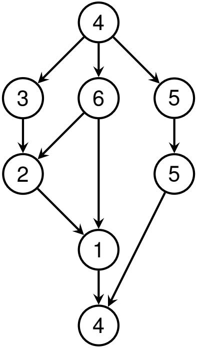
Total time to execute everything on a single processor.

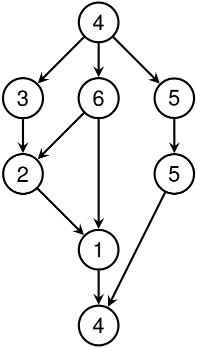


Performance Measures

$\sum = 30$

Work

Total time to execute everything on a single processor.



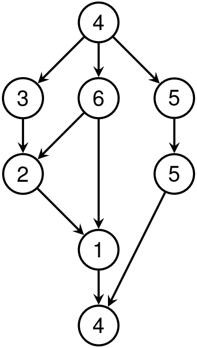
Performance Measures

Work

Total time to execute everything on a single processor.

Span

Longest time to execute the threads along any path.



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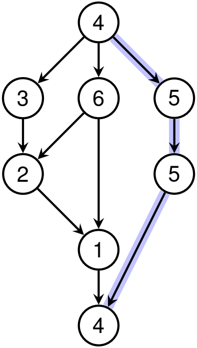
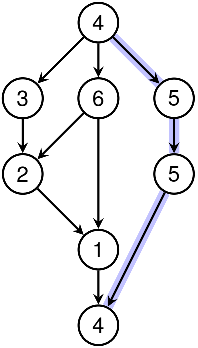


Performance Measures

$\sum = 18$



Work

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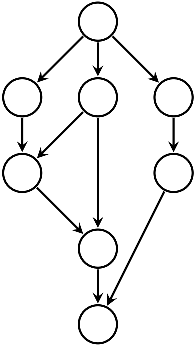


Performance Measures

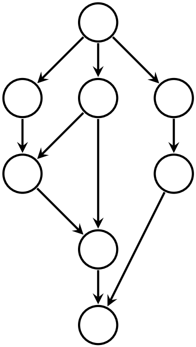
Work

Total time to execute everything on a single processor.

Span

Longest time to execute the threads along any path.

If each thread takes unit time, span is the length of the critical path.



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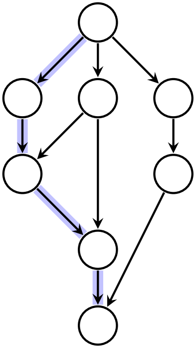

Performance Measures

Work

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Span

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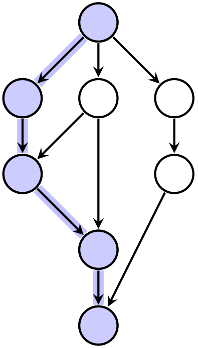
If each thread takes unit time, span is the length of the critical path.

$\#\text{nodes} = 5$



Work Law and Span Law

Work Law and Span Law

- $T_1 =$ work, $T_\infty =$ span

Work Law and Span Law

- $T_1 =$ work, $T_\infty =$ span
- $P =$ number of (identical) processors
- $T_P =$ running time on P processors

Work Law and Span Law

- $T_1 =$ work, $T_\infty =$ span
- $P =$ number of (identical) processors
- $T_P =$ running time on P processors

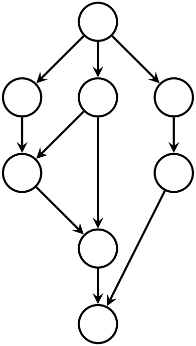
Running time actually also depends on scheduler etc.!

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

Work Law

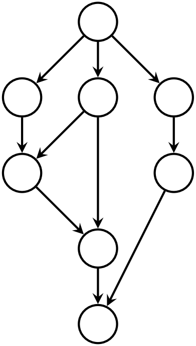
$$T_P \geq \frac{T_1}{P}$$



Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_1 = 8,\ P = 2$



Work Law

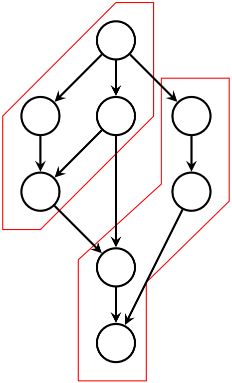
$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_1 = 8$, $P = 2$



Work Law

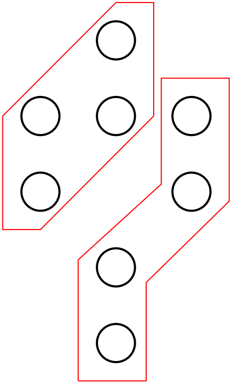
$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_1 = 8, P = 2$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

Work Law and Span Law

- T_1 = work, T_∞ = span
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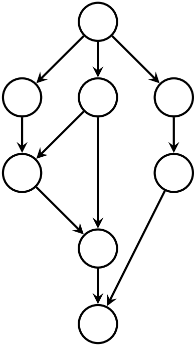
Time on P processors can't be shorter than if all work all time



Work Law and Span Law

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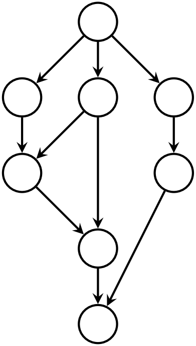
Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

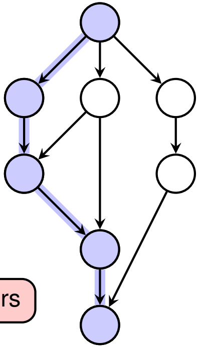
$$T_P \geq T_\infty$$



Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_\infty = 5$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

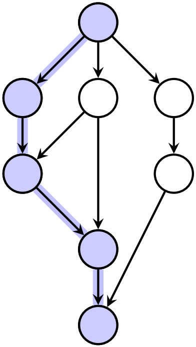
$$T_P \geq T_\infty$$

Time on P processors can't be shorter than time on ∞ processors

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$$T_\infty = 5$$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$

Work Law and Span Law

- T_1 = work, T_∞ = span
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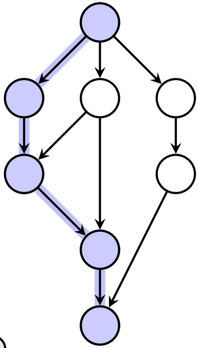


Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

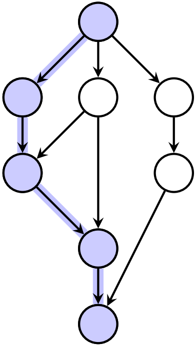
- Speed-Up: $\frac{T_1}{T_P}$

Maximum Speed-Up bounded by P!

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_\infty = 5$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$
- Parallelism: $\frac{T_1}{T_\infty}$

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
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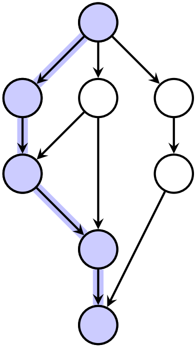
$T_\infty = 5$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$
- Parallelism: $\frac{T_1}{T_\infty}$

Maximum Speed-Up for ∞ processors!

Outline

Introduction

Serial Matrix Multiplication

Digression: Multithreading

Multithreaded Matrix Multiplication

Warmup: Matrix Vector Multiplication

Remember: Multiplying an $n \times n$ matrix $A = (a_{ij})$ and n-vector $x = (x_j)$ yields an n-vector $y = (y_i)$ given by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for } i = 1, 2, \ldots, n.$$

Warmup: Matrix Vector Multiplication

Remember: Multiplying an $n \times n$ matrix $A = (a_{ij})$ and n-vector $x = (x_j)$ yields an n-vector $y = (y_i)$ given by

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```
MAT-VEC(A, x)
n = A.rows
let y be a new vector of length n
parallel for i = 1 to n
    y_i = 0
parallel for i = 1 to n
    for j = 1 to n
        y_i = y_i + a_ij x_j
return y
```

Warmup: Matrix Vector Multiplication

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return y
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The **parallel for**-loops can be used since different entries of y can be computed concurrently.

Warmup: Matrix Vector Multiplication

Remember: Multiplying an $n \times n$ matrix $A = (a_{ij})$ and n-vector $x = (x_j)$ yields an n-vector $y = (y_i)$ given by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for } i = 1, 2, \ldots, n.$$

MAT-VEC(A, x)

$n = A.rows$
let y be a new vector of length n
parallel for $i = 1$ **to** n
    $y_i = 0$
parallel for $i = 1$ **to** n
    for $j = 1$ **to** n
        $y_i = y_i + a_{ij} x_j$
return y

The `parallel for`-loops can be used since different entries of y can be computed concurrently.

How can a compiler implement the `parallel for`-loop?

Implementing `parallel for` based on Divide-and-Conquer

MAT-VEC-MAIN-LOOP(A, x, y, n, i, i')

```
if i == i'
    for j = 1 to n
        y_i = y_i + a_ij x_j
else mid = ⌊(i + i')/2⌋
    spawn MAT-VEC-MAIN-LOOP(A, x, y, n, i, mid)
    MAT-VEC-MAIN-LOOP(A, x, y, n, mid + 1, i')
    sync
```

MAT-VEC(A, x)

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n = A.rows
let y be a new vector of length n
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```
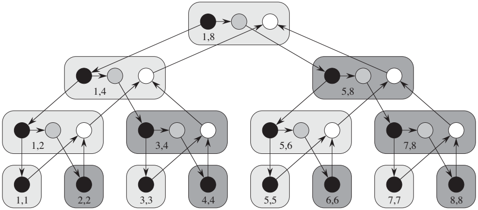

Implementing `parallel for` based on Divide-and-Conquer



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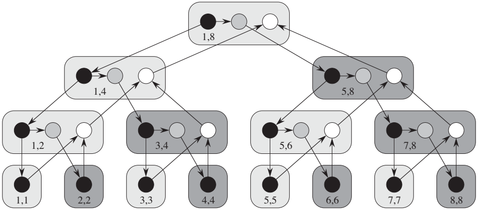

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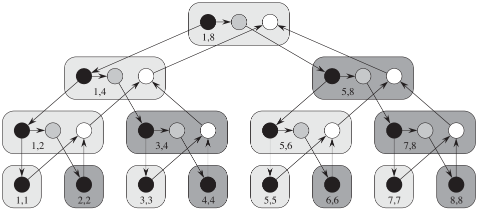
    for j = 1 to n
        y_i = y_i + a_ij x_j
return y
```

$$T_1(n) =$$

Implementing `parallel for` based on Divide-and-Conquer

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return y
```
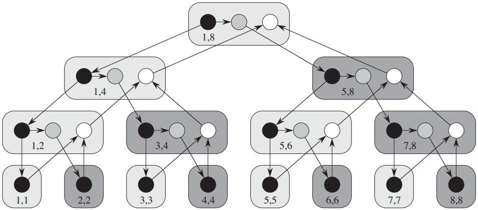

$T_1(n) =$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

Implementing `parallel for` based on Divide-and-Conquer



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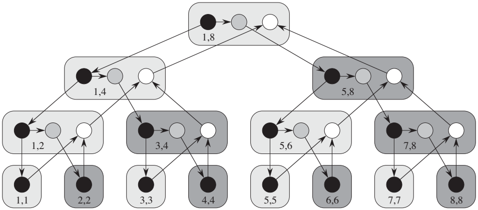
return y
```

$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

Implementing `parallel for` based on Divide-and-Conquer

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    for j = 1 to n
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return y
```

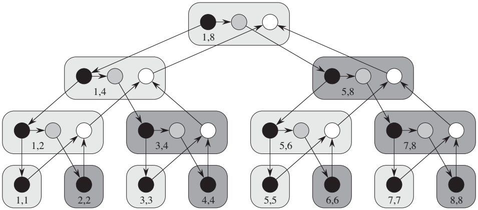
$T_1(n) = \Theta(n^2)$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$T_\infty(n) =$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Implementing `parallel for` based on Divide-and-Conquer



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  spawn MAT-VEC-MAIN-LOOP(A, x, y, n, i, mid)
  MAT-VEC-MAIN-LOOP$(A, x, y, n, mid + 1, i')$
  sync

MAT-VEC(A, x)

$n = A.rows$
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  for $j = 1$ **to** n
    $y_i = y_i + a_{ij} x_j$
return y

$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

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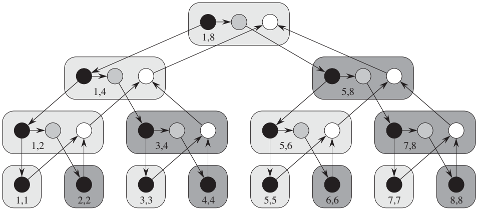


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parallel for i = 1 to n
    y_i = 0
parallel for i = 1 to n
    for j = 1 to n
        y_i = y_i + a_ij x_j
return y
```

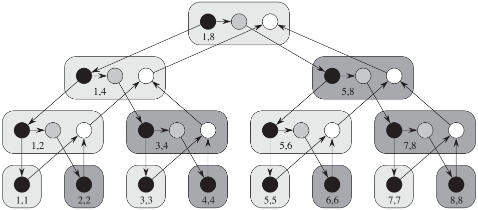
$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) = \Theta(\log n) + \max_{1 \le i \le n} \text{iter}(n)$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Implementing `parallel for` based on Divide-and-Conquer



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    MAT-VEC-MAIN-LOOP(A, x, y, n, mid + 1, i')
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return y
```

$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) = \Theta(\log n) + \max_{1 \leq i \leq n} \text{iter}(n) = \Theta(n).$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Naive Algorithm in Parallel

```
P-SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
parallel for i = 1 to n
    parallel for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

Naive Algorithm in Parallel

P-SQUARE-MATRIX-MULTIPLY(A, B)

```
n = A.rows
let C be a new n × n matrix
parallel for i = 1 to n
    parallel for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

With a more careful implementation, $T_\infty(n) = O(\log n)$ (CLRS, Exercise 27.2-3)

P-SQUARE-MATRIX-MULTIPLY(A, B) has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(n)$.

The first two nested for-loops parallelise perfectly.

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)
n = A.rows
if n == 1
    c_11 = a_11 b_11
else let T be a new n × n matrix
    partition A, B, C, and T into n/2 × n/2 submatrices
        A_11, A_12, A_21, A_22; B_11, B_12, B_21, B_22; C_11, C_12, C_21, C_22;
        and T_11, T_12, T_21, T_22; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_11, A_11, B_11)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_12, A_11, B_12)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_21, A_21, B_11)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_22, A_21, B_12)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_11, A_12, B_21)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_12, A_12, B_22)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_21, A_22, B_21)
    P-MATRIX-MULTIPLY-RECURSIVE(T_22, A_22, B_22)
    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            c_ij = c_ij + t_ij
```

The Simple Divide&Conquer Approach in Parallel

P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)

$n = A.rows$
if $n == 1$
    $c_{11} = a_{11}b_{11}$
else let T be a new $n \times n$ matrix
    partition A, B, C, and T into $n/2 \times n/2$ submatrices
        $A_{11}, A_{12}, A_{21}, A_{22}$; $B_{11}, B_{12}, B_{21}, B_{22}$; $C_{11}, C_{12}, C_{21}, C_{22}$;
        and $T_{11}, T_{12}, T_{21}, T_{22}$; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_{11}, A_{11}, B_{11})
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    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_{21}, A_{22}, B_{21})
    P-MATRIX-MULTIPLY-RECURSIVE(T_{22}, A_{22}, B_{22})
    sync
    parallel for $i = 1$ **to** n
        parallel for $j = 1$ **to** n
            $c_{ij} = c_{ij} + t_{ij}$

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) =$

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)
n = A.rows
if n == 1
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    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            c_ij = c_ij + t_ij
```

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) =$

$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$

The Simple Divide&Conquer Approach in Parallel

P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)

$n = A.rows$
if $n == 1$
    $c_{11} = a_{11} b_{11}$
else let T be a new $n \times n$ matrix
    partition A, B, C, and T into $n/2 \times n/2$ submatrices
        $A_{11}, A_{12}, A_{21}, A_{22}$; $B_{11}, B_{12}, B_{21}, B_{22}$; $C_{11}, C_{12}, C_{21}, C_{22}$;
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    P-MATRIX-MULTIPLY-RECURSIVE(T_{22}, A_{22}, B_{22})
    sync
    parallel for $i = 1$ **to** n
        parallel for $j = 1$ **to** n
            $c_{ij} = c_{ij} + t_{ij}$

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(\log^2 n)$.

$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$

Strassen's Algorithm in Parallel

Strassen's Algorithm (parallelised)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

Strassen's Algorithm in Parallel

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   This step takes $\Theta(1)$ work and span by index calculations.

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1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

   This step takes $\Theta(1)$ work and span by index calculations.

2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.

Strassen's Algorithm in Parallel

Strassen's Algorithm (parallelised)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

   This step takes $\Theta(1)$ work and span by index calculations.

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   Can create all 10 matrices with $\Theta(n^2)$ work and $\Theta(\log n)$ span using doubly nested **`parallel for`** loops.

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   Can create all 10 matrices with $\Theta(n^2)$ work and $\Theta(\log n)$ span using doubly nested **`parallel for`** loops.

3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$

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   Recursively **`spawn`** the computation of the seven products.

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   Recursively **`spawn`** the computation of the seven products.

4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

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1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

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   Using doubly nested **`parallel for`** this takes $\Theta(n^2)$ work and $\Theta(\log n)$ span.

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1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

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$$T_1(n) = \Theta(n^{\log 7})$$

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$$T_1(n) = \Theta(n^{\log 7})$$
$$T_\infty(n) = \Theta(\log^2 n)$$

III. Linear Programming

Thomas Sauerwald

Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Finding an Initial Solution

Introduction

Linear Programming (informal definition)

- maximize or minimize an objective, given limited resources and competing constraint
- constraints are specified as (in)equalities

Introduction

Linear Programming (informal definition)

- maximize or minimize an objective, given limited resources and competing constraint
- constraints are specified as (in)equalities

Example: Political Advertising

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- Imagine you are a politician trying to win an election

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- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters

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- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters
- Aim: at least half of the registered voters in each of the three regions should vote for you

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- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters
- Aim: at least half of the registered voters in each of the three regions should vote for you
- Possible Actions: Advertise on one of the primary issues which are (i) building more roads, (ii) gun control, (iii) farm subsidies and (iv) a gasoline tax dedicated to improve public transit.

Political Advertising Continued

policy	urban	suburban	rural
build roads	-2	5	3
gun control	8	2	-5
farm subsidies	0	0	10
gasoline tax	10	0	-2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

Political Advertising Continued

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- Possible Solution:
  - $20,000 on advertising to building roads
  - $0 on advertising to gun control
  - $4,000 on advertising to farm subsidies
  - $9,000 on advertising to a gasoline tax

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What is the best possible strategy?

Towards a Linear Program

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The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

- $x_1 =$ number of thousands of dollars spent on advertising on building roads
- $x_2 =$ number of thousands of dollars spent on advertising on gun control
- $x_3 =$ number of thousands of dollars spent on advertising on farm subsidies
- $x_4 =$ number of thousands of dollars spent on advertising on gasoline tax

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Constraints:

Towards a Linear Program

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Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$

Towards a Linear Program

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- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$

Towards a Linear Program

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Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
- $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

Towards a Linear Program

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Objective: Minimize $x_1 + x_2 + x_3 + x_4$

The Linear Program

Linear Program for the Advertising Problem

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & \multicolumn{5}{l}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}
$$

The Linear Program

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$$
\begin{array}{lrcrcrcrcr}
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\text{subject to} & & & & & & & & & \\
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& & & x_1, x_2, x_3, x_4 & & & & & \geq & 0
\end{array}
$$

The solution of this linear program yields the optimal advertising strategy.

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\begin{array}{lrcrcrcrcr}
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& & & x_1, x_2, x_3, x_4 & & & & & \geq & 0
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- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

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- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \gtrless b$

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Linear Program for the Advertising Problem

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Linear Constraints

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcl}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
& -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
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& & & x_1, x_2, x_3, x_4 & & & & & \geq & 0
\end{array}$$

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Linear Constraints

- Linear-Progamming Problem: either minimize or maximize a linear function subject to a set of linear constraints

A Small(er) Example

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}
$$

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\text{maximize} & x_1 & + & x_2 & & \\
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Any setting of x_1 and x_2 satisfying all constraints is a feasible solution

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\begin{array}{lrcrcr}
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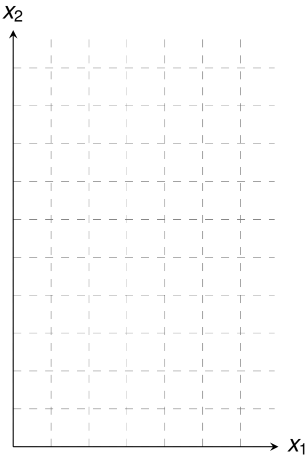

A Small(er) Example

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\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
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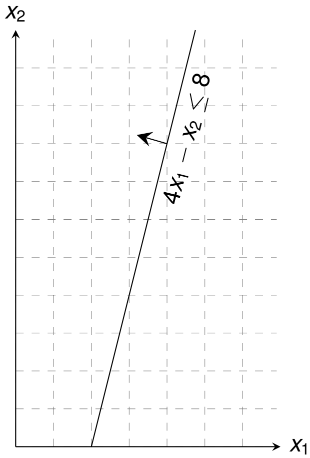

A Small(er) Example

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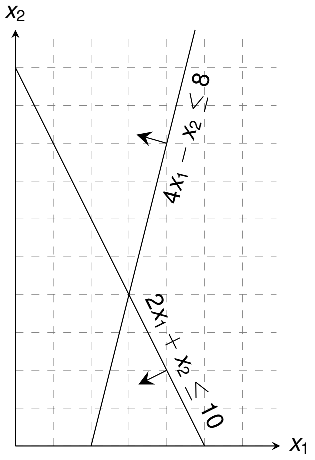

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\text{maximize} & x_1 & + & x_2 & & \\
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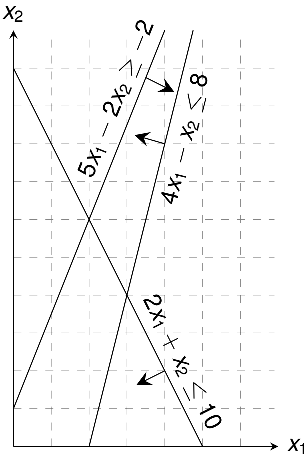

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\text{maximize} & x_1 & + & x_2 & & \\
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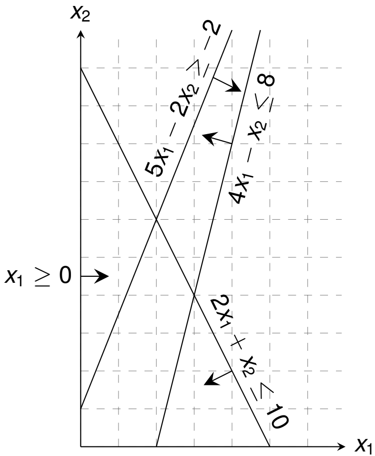

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\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
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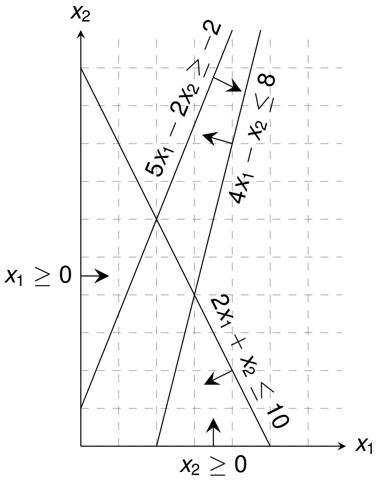

A Small(er) Example

$$
\begin{array}{llrcl}
\text{maximize} & x_1 + x_2 & & & \\
\text{subject to} & & & & \\
& 4x_1 - x_2 & \leq & 8 \\
& 2x_1 + x_2 & \leq & 10 \\
& 5x_1 - 2x_2 & \geq & -2 \\
& x_1, x_2 & \geq & 0
\end{array}
$$

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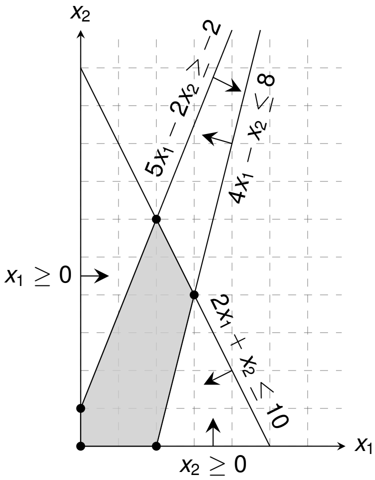
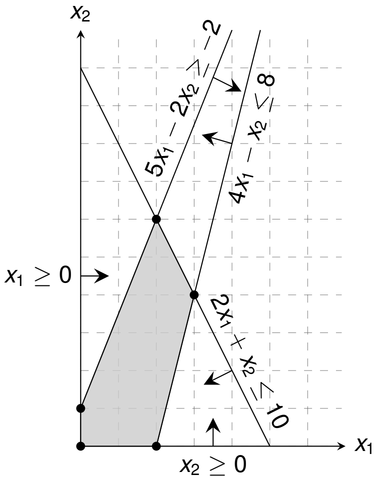


A Small(er) Example

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



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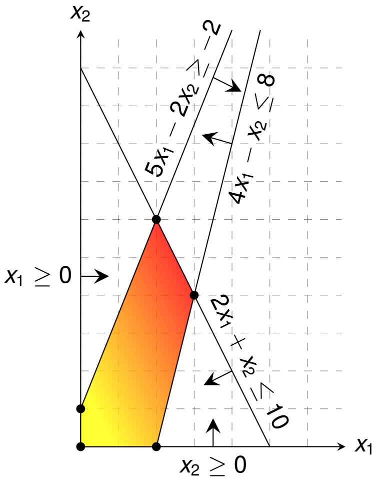
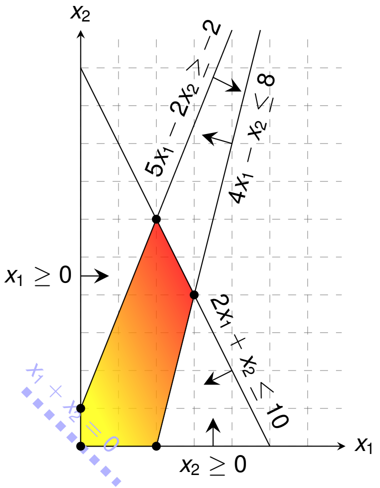


Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

A Small(er) Example

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1, x_2 & & & \geq & 0
\end{array}$$

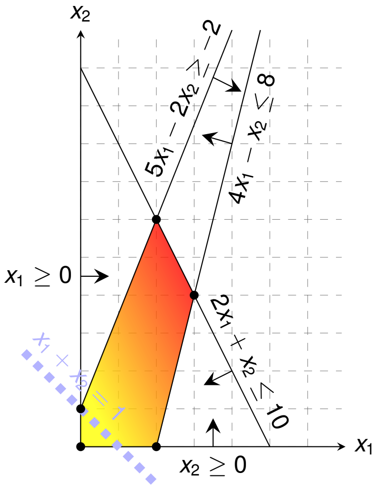
Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1, x_2 & & & \geq & 0
\end{array}$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



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$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
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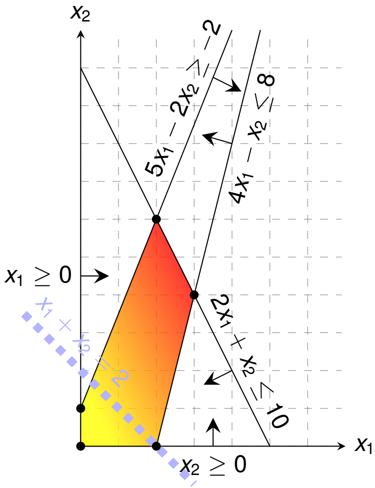

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$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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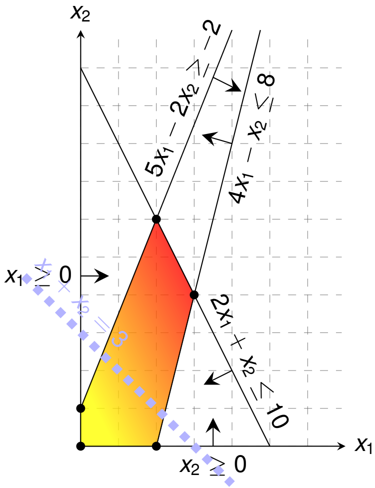
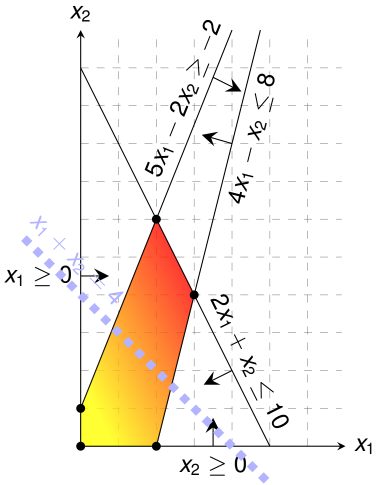


A Small(er) Example

$$\begin{array}{llcl}
\text{maximize} & x_1 + x_2 & & \\
\text{subject to} & & & \\
 & 4x_1 - x_2 & \leq & 8 \\
 & 2x_1 + x_2 & \leq & 10 \\
 & 5x_1 - 2x_2 & \geq & -2 \\
 & x_1, x_2 & \geq & 0
\end{array}$$



Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

A Small(er) Example

$$
\begin{array}{llllll}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
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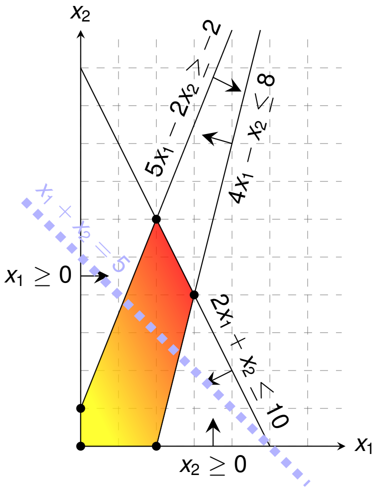

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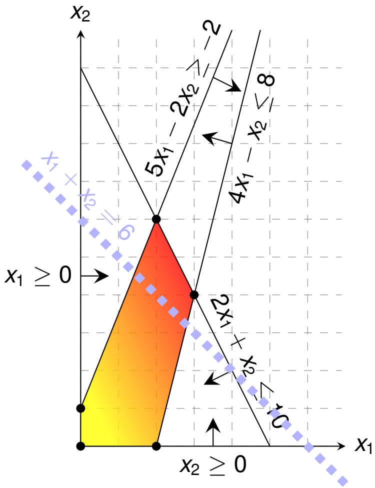

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
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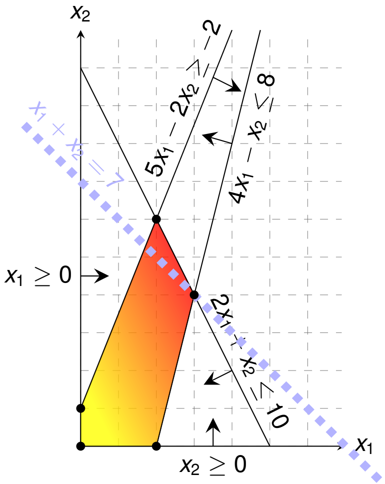

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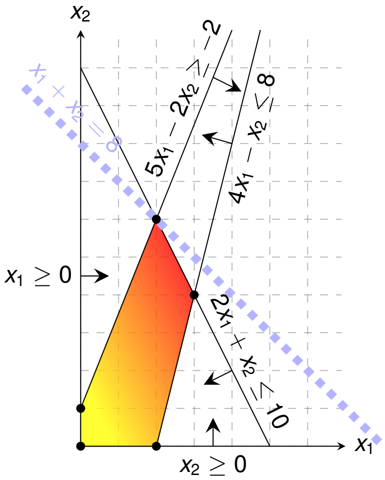

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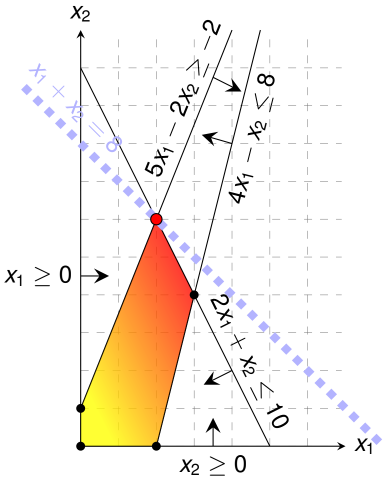

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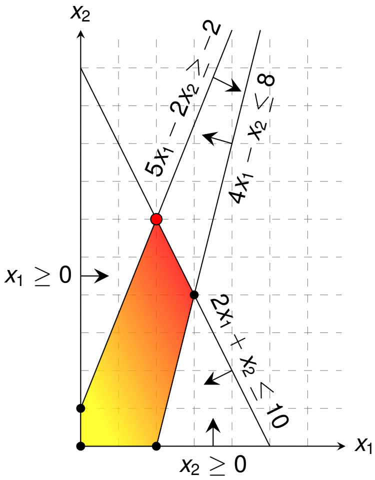

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$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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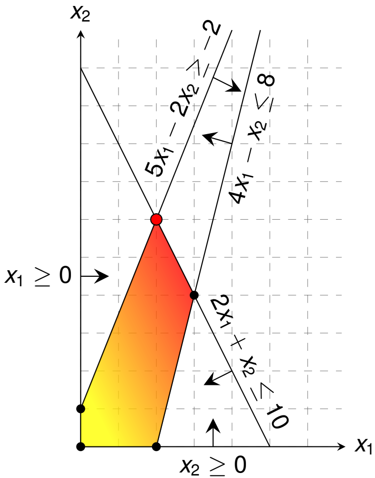

While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Finding an Initial Solution

Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

maximize $\sum_{j=1}^{n} c_j x_j$ Objective Function

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$n + m$ Constraints

$$\begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, 2, \ldots, n \end{cases}$$

Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$$\underbrace{\begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, 2, \ldots, n \end{cases}}_{n + m \text{ Constraints}}$$

Non-Negativity Constraints

Standard and Slack Forms

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$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

$n + m$ Constraints

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

$$\text{maximize} \quad c^T x$$

Inner product of two vectors

subject to

$$Ax \leq b$$

Matrix-vector product

$$x \geq 0$$

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimization rather than maximization.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

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Equivalence: a correspondence (not necessarily a bijection) between solutions so that their objective values are identical.

When switching from maximization to minimization, sign of objective value changes.

Converting into Standard Form (1/5)

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$$\begin{array}{lrcrcl} \boxed{\text{minimize} \quad -2x_1 + 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

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1. The objective might be a minimization rather than maximization.

$$\boxed{\text{minimize} \quad -2x_1 + 3x_2}$$
subject to
$$\begin{aligned} x_1 + x_2 &= 7 \\ x_1 - 2x_2 &\leq 4 \\ x_1 &\geq 0 \end{aligned}$$

Negate objective function

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1. The objective might be a minimization rather than maximization.

$$\begin{array}{llcrcl} \boxed{\text{minimize} \quad -2x_1 + 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

Negate objective function

$$\begin{array}{llcrcl} \boxed{\text{maximize} \quad 2x_1 - 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

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$$
\begin{array}{lrcrcl}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& \boxed{x_1} & & & \geq & 0
\end{array}
$$

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$$
\begin{array}{llcrcl}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
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\end{array}
$$

Replace x_2 by two non-negative variables x_2' and x_2''

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$$
\begin{array}{llcllcl}
\text{maximize} & 2x_1 & - & 3x_2 & & & \\
\text{subject to} & & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& \boxed{x_1} & & & \geq & 0
\end{array}
$$

Replace x_2 by two non-negative variables x_2' and x_2''

$$
\begin{array}{llclcl}
\text{maximize} & 2x_1 & - & \boxed{3x_2' + 3x_2''} & & \\
\text{subject to} & & & & & \\
& x_1 & + & \boxed{x_2' - x_2''} & = & 7 \\
& x_1 & - & \boxed{2x_2' + 2x_2''} & \leq & 4 \\
& \boxed{x_1, x_2', x_2''} & & & \geq & 0
\end{array}
$$

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$$
\begin{array}{llcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& \boxed{x_1 + x_2' - x_2'' = 7} & & & & & & \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & & & & & \geq & 0
\end{array}
$$

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$$
\begin{array}{llcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& \boxed{x_1 & + & x_2' & - & x_2'' & = & 7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & x_1, x_2', x_2'' & & & & \geq & 0
\end{array}
$$

Replace each equality by two inequalities.

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$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& \boxed{x_1} & \boxed{+} & \boxed{x_2'} & \boxed{-} & \boxed{x_2''} & \boxed{=} & \boxed{7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & & & x_1, x_2', x_2'' & \geq & 0
\end{array}$$

Replace each equality by two inequalities.

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & & & x_1, x_2', x_2'' & \geq & 0
\end{array}$$

Converting into Standard Form (4/5)

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\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& \boxed{x_1} & \boxed{+} & \boxed{x_2'} & \boxed{-} & \boxed{x_2''} & \boxed{\geq} & \boxed{7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}$$

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$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & \boxed{x_1} & \boxed{+} & \boxed{x_2'} & \boxed{-} & \boxed{x_2''} & \boxed{\geq} & \boxed{7} \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, x_2', x_2'' & & & & & \geq & 0
\end{array}
$$

Negate respective inequalities.

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$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& \boxed{x_1} & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
$$

Negate respective inequalities.

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

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\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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For the simplex algorithm, it is more convenient to work with equality constraints.

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Introducing Slack Variables

Converting Standard Form into Slack Form (1/3)

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For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable s by

Converting Standard Form into Slack Form (1/3)

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- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable s by

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

Converting Standard Form into Slack Form (1/3)

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$$s \geq 0.$$

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$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the ith inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

$$\Big\downarrow \text{ Introduce slack variables}$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{llllllll}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}$$

Introduce slack variables

subject to

$$x_4 = 7 - x_1 - x_2 + x_3$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

Introduce slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
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\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
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$$

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\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
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Introduce slack variables

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& & & x_1, x_2, x_3, x_4, x_5, x_6 & & \geq & & 0 & &
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& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{4}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & & \geq & 0 & &
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrrrrrrrrr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrrrrrrrr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rrrrrrrrr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrrrrrrrr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{4}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{array}{rrrrrrrrr}
\boxed{z} & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

This is called slack form.

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
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x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

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x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Basic and Non-Basic Variables

$$\begin{array}{rclclclcl} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
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\end{array}
$$

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and all variables are non-negative.

Variables/Coefficients on the right hand side are indexed by B and N.

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
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\end{aligned}
$$

Slack Form Notation

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$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
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\end{aligned}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

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$$
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\end{aligned}
$$

Slack Form Notation

- $B = \{1,2,4\}$, $N = \{3,5,6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

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z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
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\end{aligned}$$

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- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

Slack Form (Example)

$$\begin{array}{rcrcrcrcr} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

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- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$

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$$
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- $v = 28$

The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

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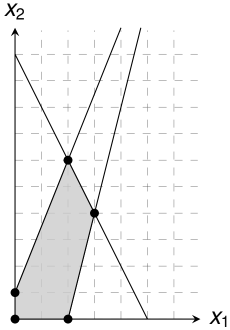
Theorem

If the slack form has an optimal solution, one of them occurs at a vertex.



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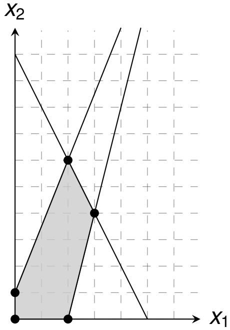
Proof Sketch (informal and non-examinable):

- Rewrite LP s.t. $Ax = b$. Let x be optimal but not a vertex



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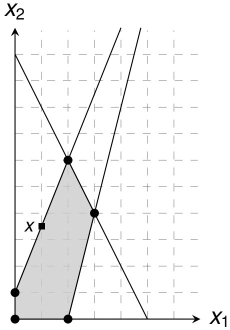

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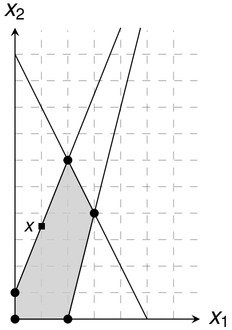

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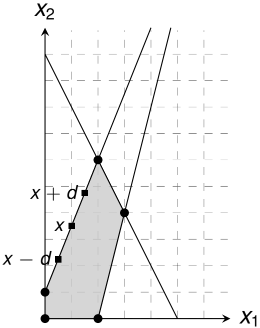

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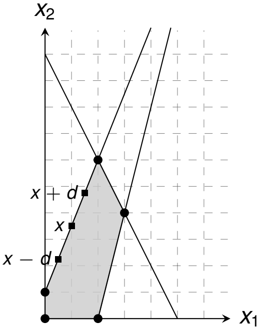

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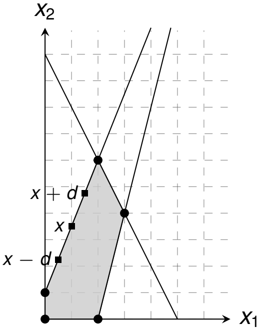

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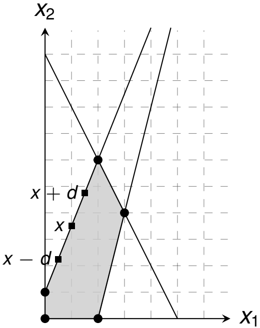

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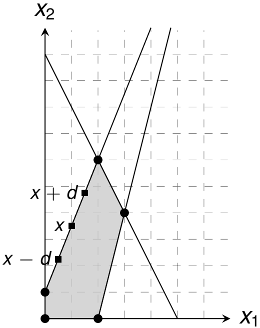

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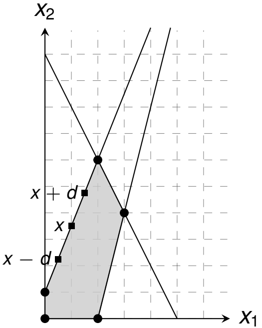

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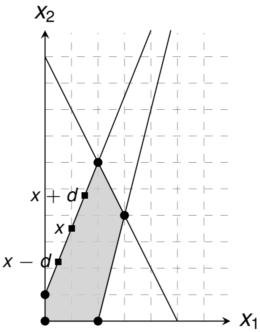

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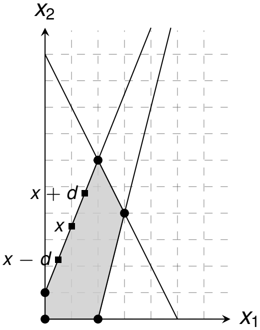

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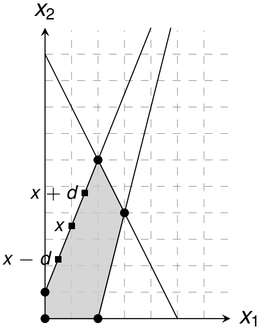

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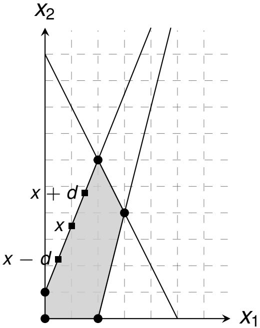

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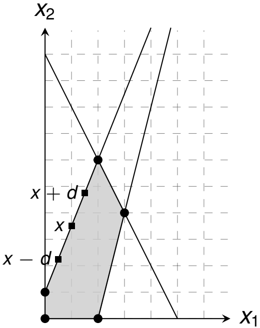

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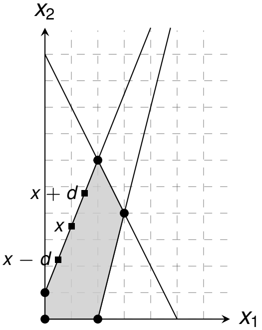
  $\Rightarrow$ This contradicts the assumption that there exists an optimal solution.



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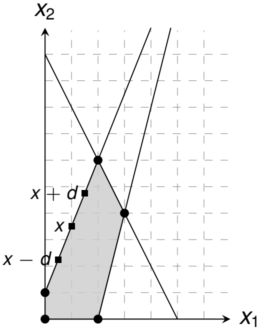

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Finding an Initial Solution

Shortest Paths

Single-Pair Shortest Path Problem

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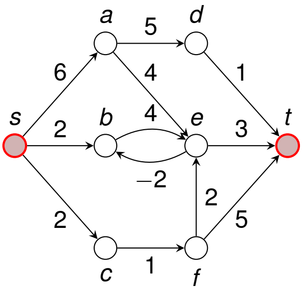

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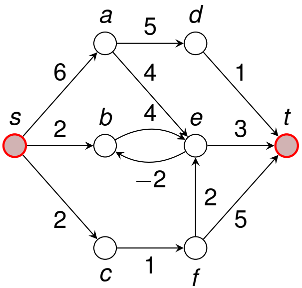

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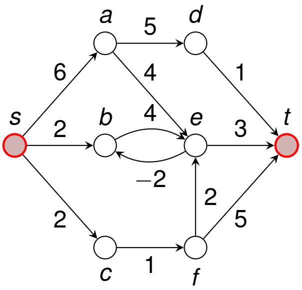

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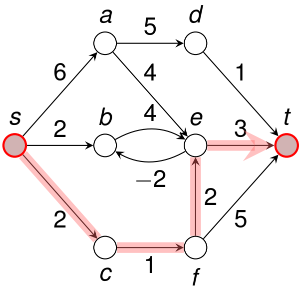

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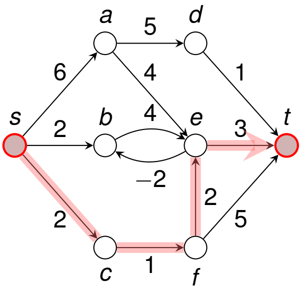

Shortest Paths as LP

subject to

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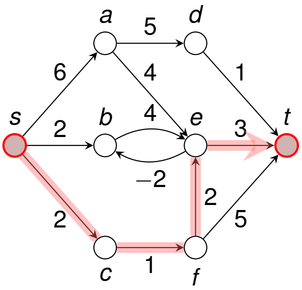

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$$\begin{aligned} d_v &\leq d_u + w(u,v) \quad \text{for each edge } (u,v) \in E, \\ d_s &= 0. \end{aligned}$$

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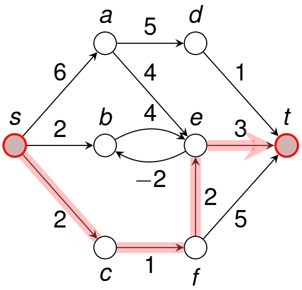

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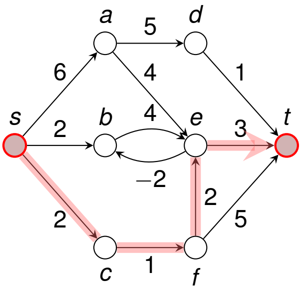

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this is a maximization problem!

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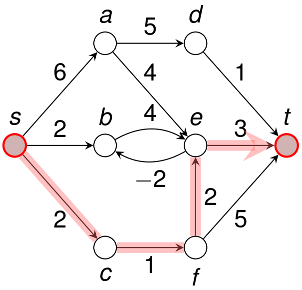

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Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

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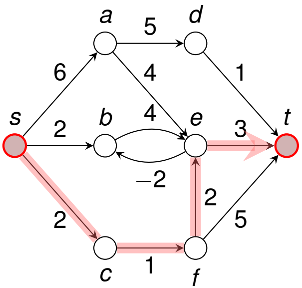

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Solution $\overline{d}$ satisfies $\overline{d}_v = \min_{u:\,(u,v)\in E} \left\{ \overline{d}_u + w(u,v) \right\}$

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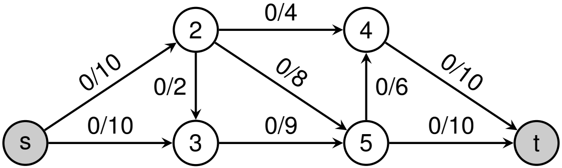

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- Given: directed graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$
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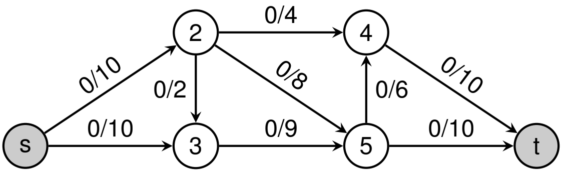

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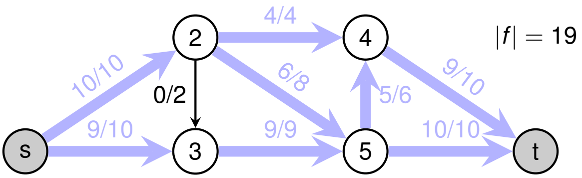

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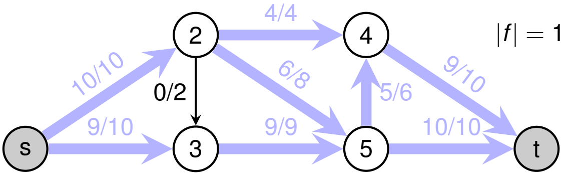

$|f| = 19$

Maximum Flow as LP

$$
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\text{subject to} & & & & \\
& f_{uv} & \leq & c(u,v) & \text{for each } u, v \in V, \\
& \sum_{v\in V} f_{vu} & = & \sum_{v\in V} f_{uv} & \text{for each } u \in V \setminus \{s,t\}, \\
& f_{uv} & \geq & 0 & \text{for each } u, v \in V.
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Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- Given: directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of d units

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- Goal: Find a flow $f : V \times V \to \mathbb{R}$ from s to t with $|f| = d$ while minimising the total cost $\sum_{(u,v)\in E} a(u, v) f_{uv}$ incurrred by the flow.

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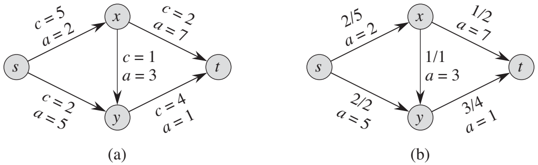

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

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Optimal Solution with total cost:
$\sum_{(u,v)\in E} a(u,v) f_{uv} = (2\cdot 2)+(5\cdot 2)+(3\cdot 1)+(7\cdot 1)+(1\cdot 3) = 27$

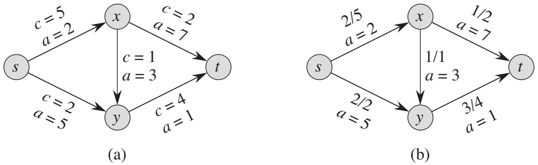

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

Minimum-Cost Flow as a LP

Minimum Cost Flow as LP

minimize $\quad \sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$\begin{array}{rcll} f_{uv} & \leq & c(u,v) & \text{for each } u, v \in V, \\ \sum_{v\in V} f_{vu} - \sum_{v \in V} f_{uv} & = & 0 & \text{for each } u \in V \setminus \{s,t\}, \\ \sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & = & d\,, & \\ f_{uv} & \geq & 0 & \text{for each } u, v \in V. \end{array}$$

Minimum-Cost Flow as a LP

Minimum Cost Flow as LP

minimize $\sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$
\begin{aligned}
f_{uv} &\leq c(u,v) && \text{for each } u,v \in V, \\
\sum_{v\in V} f_{vu} - \sum_{v \in V} f_{uv} &= 0 && \text{for each } u \in V \setminus \{s,t\}, \\
\sum_{v\in V} f_{sv} - \sum_{v \in V} f_{vs} &= d\ , && \\
f_{uv} &\geq 0 && \text{for each } u,v \in V.
\end{aligned}
$$

Real power of Linear Programming comes from the ability to solve **new problems**!

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Finding an Initial Solution

Simplex Algorithm: Introduction

Simplex Algorithm

- classical method for solving linear programs (Dantzig, 1947)
- usually fast in practice although worst-case runtime not polynomial
- iterative procedure somewhat similar to Gaussian elimination

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Basic Idea:

- Each iteration corresponds to a “basic solution” of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints
- Each iteration converts one slack form into an equivalent one while the objective value will not decrease
- Conversion (“pivoting”) is achieved by switching the roles of one basic and one non-basic variable

Simplex Algorithm: Introduction

Simplex Algorithm

- classical method for solving linear programs (Dantzig, 1947)
- usually fast in practice although worst-case runtime not polynomial
- iterative procedure somewhat similar to Gaussian elimination

Basic Idea:

- Each iteration corresponds to a "basic solution" of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints
- Each iteration converts one slack form into an equivalent one while the objective value will not decrease

In that sense, it is a greedy algorithm.

- Conversion ("pivoting") is achieved by switching the roles of one basic and one non-basic variable

Extended Example: Conversion into Slack Form

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & \multicolumn{3}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Extended Example: Conversion into Slack Form

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Conversion into slack form

Extended Example: Conversion into Slack Form

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\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Conversion into slack form

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Extended Example: Iteration 1

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
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\end{array}
$$

Extended Example: Iteration 1

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

Extended Example: Iteration 1

$$\begin{array}{rclcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

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This basic solution is **feasible**

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$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

This basic solution is **feasible**

Objective value is 0.

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

This basic solution is **feasible**

Objective value is 0.

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Increasing the value of x_1 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_1.

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
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\end{array}
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The third constraint is the tightest and limits how much we can increase x_1.

Switch roles of x_1 and x_6:

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Increasing the value of x_1 would increase the objective value.

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\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
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\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_1.

Switch roles of x_1 and x_6:

- Solving for x_1 yields:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
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The third constraint is the tightest and limits how much we can increase x_1.

Switch roles of x_1 and x_6:

- Solving for x_1 yields:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$

- Substitute this into x_1 in the other three equations

Extended Example: Iteration 2

$$
\begin{array}{rclcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

Extended Example: Iteration 2

$$
\begin{array}{rcrcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (9, 0, 0, 21, 6, 0)$ with objective value 27

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (9, 0, 0, 21, 6, 0)$ with objective value 27

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
\boxed{x_5} & \boxed{=} & \boxed{6} & \boxed{-} & \boxed{\frac{3x_2}{2}} & \boxed{-} & \boxed{4x_3} & \boxed{+} & \boxed{\frac{x_6}{2}}
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{lllllllll}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{lclclclcl}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

- Solving for x_3 yields:

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.$$

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{lcrcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

- Solving for x_3 yields:

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.$$

- Substitute this into x_3 in the other three equations

Extended Example: Iteration 3

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

Extended Example: Iteration 3

$$
\begin{array}{rclclclcl}
z & = & \dfrac{111}{4} & + & \dfrac{x_2}{16} & - & \dfrac{x_5}{8} & - & \dfrac{11x_6}{16} \\
x_1 & = & \dfrac{33}{4} & - & \dfrac{x_2}{16} & + & \dfrac{x_5}{8} & - & \dfrac{5x_6}{16} \\
x_3 & = & \dfrac{3}{2} & - & \dfrac{3x_2}{8} & - & \dfrac{x_5}{4} & + & \dfrac{x_6}{8} \\
x_4 & = & \dfrac{69}{4} & + & \dfrac{3x_2}{16} & + & \dfrac{5x_5}{8} & - & \dfrac{x_6}{16}
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$\begin{array}{rclcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_4}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

The second constraint is the tightest and limits how much we can increase x_2.

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$\begin{array}{rclllllll} z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\ x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\ x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\ x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16} \end{array}$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

- Solving for x_2 yields:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

- Solving for x_2 yields:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$

- Substitute this into x_2 in the other three equations

Extended Example: Iteration 4

$$
\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Extended Example: Iteration 4

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (8, 4, 0, 18, 0, 0)$ with objective value 28

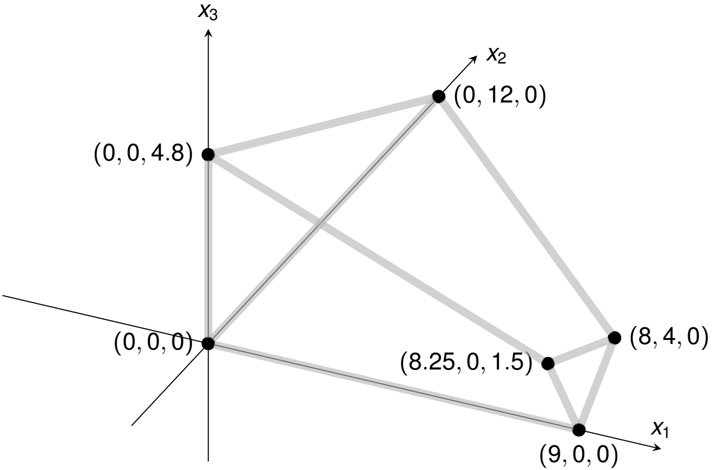
Extended Example: Iteration 4

All coefficients are negative, and hence this basic solution is **optimal**!

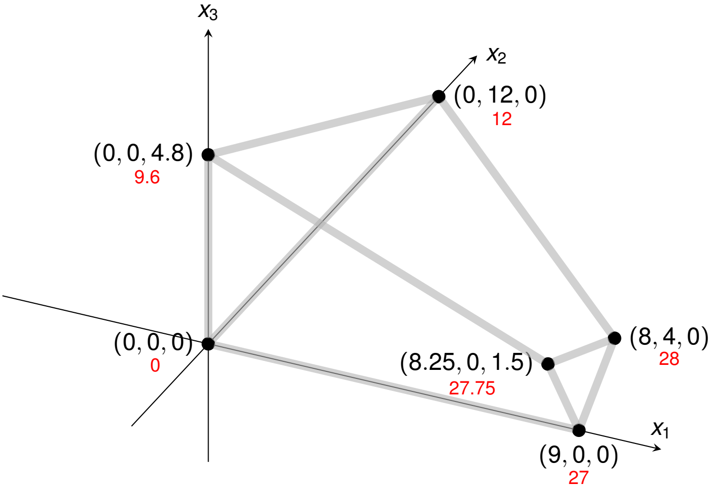
$$\begin{array}{rclclclcl}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (8, 4, 0, 18, 0, 0)$ with objective value 28

Extended Example: Visualization of SIMPLEX

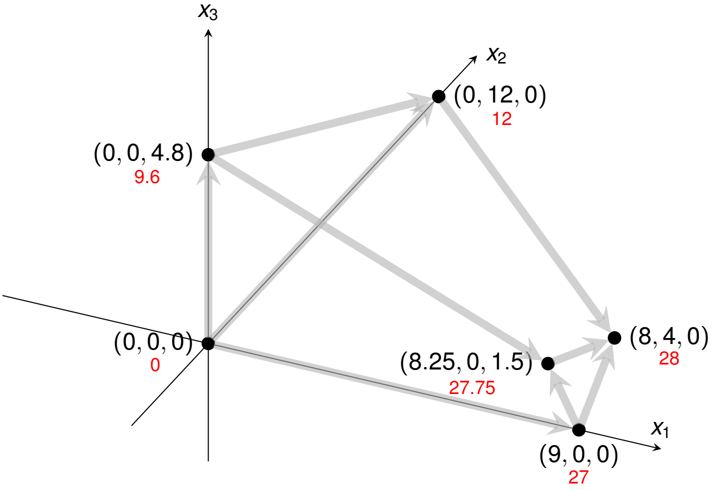


Extended Example: Visualization of SIMPLEX



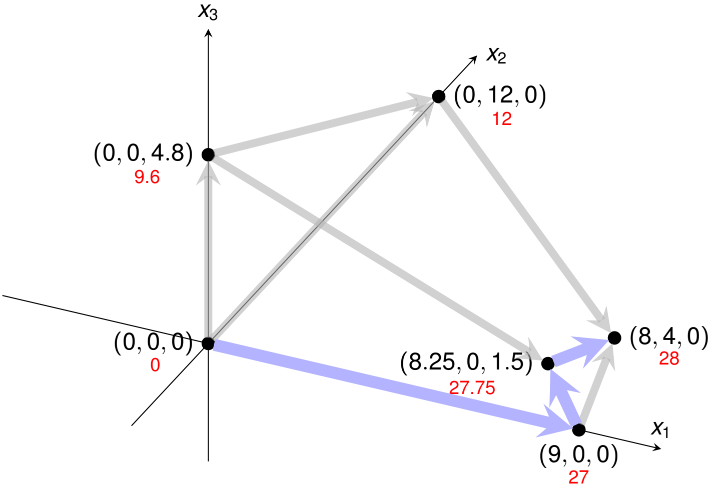
Exercise: How many basic solutions (including non-feasible ones) are there?

Extended Example: Visualization of SIMPLEX



Exercise: How many basic solutions (including non-feasible ones) are there?

Extended Example: Visualization of SIMPLEX



Exercise: How many basic solutions (including non-feasible ones) are there?

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_2 and x_5

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_2 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}
$$

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{lllllllll}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

$\big\downarrow$ Switch roles of x_2 and x_5

$$
\begin{array}{lllllllll}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}
$$

$\big\downarrow$ Switch roles of x_1 and x_6

Extended Example: Alternative Runs (1/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_2 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}$$

Switch roles of x_1 and x_6

$$\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_3 and x_5

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{lcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

$$\Big\downarrow \text{ Switch roles of } x_3 \text{ and } x_5$$

$$
\begin{array}{lcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Switch roles of x_1 and x_6

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_3 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}$$

Switch roles of x_1 and x_6

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_3 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}$$

Switch roles of x_1 and x_6

Switch roles of x_2 and x_3

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Switch roles of x_1 and x_6

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Switch roles of x_2 and x_3

$$
\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

The Pivot Step Formally

$\text{PIVOT}(N, B, A, b, c, v, l, e)$

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie} \widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l = -c_e \widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

The Pivot Step Formally

PIVOT(N, B, A, b, c, v, l, e)

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l = -c_e \widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Rewrite "tight" equation for enterring variable x_e.

The Pivot Step Formally

PIVOT(N, B, A, b, c, v, l, e)

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e\widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e\widehat{a}_{ej}$
$\widehat{c}_l = -c_e\widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Rewrite "tight" equation for enterring variable x_e.

Substituting x_e into other equations.

The Pivot Step Formally

$\text{PIVOT}(N, B, A, b, c, v, l, e)$

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$

Rewrite "tight" equation for enterring variable x_e.

// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie} \widehat{a}_{el}$

Substituting x_e into other equations.

// Compute the objective function.
$\widehat{v} = v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l = -c_e \widehat{a}_{el}$

Substituting x_e into objective function.

// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

The Pivot Step Formally

PIVOT(N, B, A, b, c, v, l, e)

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
  $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1/a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
  $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
  for each $j \in N - \{e\}$
    $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
  $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
  $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l = -c_e \widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Rewrite “tight” equation for enterring variable x_e.

Substituting x_e into other equations.

Substituting x_e into objective function.

Update non-basic and basic variables

The Pivot Step Formally

PIVOT(N, B, A, b, c, v, l, e)

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
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$\widehat{v} = v + c_e \widehat{b}_e$
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$\widehat{c}_l = -c_e \widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Need that $a_{le} \neq 0$!

Rewrite “tight” equation for enterring variable x_e.

Substituting x_e into other equations.

Substituting x_e into objective function.

Update non-basic and basic variables

Effect of the Pivot Step

Lemma 29.1

Consider a call to $\textsc{Pivot}(N, B, A, b, c, v, l, e)$ in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

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1. $\overline{x}_j = 0$ for each $j \in \widehat{N}$.
2. $\overline{x}_e = b_l / a_{le}$.
3. $\overline{x}_i = b_i - a_{ie}\widehat{b}_e$ for each $i \in \widehat{B} \setminus \{e\}$.

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Proof:

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Proof:

1. holds since the basic solution always sets all non-basic variables to zero.
2. When we set each non-basic variable to 0 in a constraint
$$x_i = \widehat{b}_i - \sum_{j \in \widehat{N}} \widehat{a}_{ij} x_j,$$
we have $\overline{x}_i = \widehat{b}_i$ for each $i \in \widehat{B}$. Hence $\overline{x}_e = \widehat{b}_e = b_l / a_{le}$.
3. After the substituting in the other constraints, we have
$$\overline{x}_i = \widehat{b}_i = b_i - a_{ie}\widehat{b}_e.$$

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we have $\overline{x}_i = \widehat{b}_i$ for each $i \in \widehat{B}$. Hence $\overline{x}_e = \widehat{b}_e = b_l / a_{le}$.

3. After the substituting in the other constraints, we have

$$\overline{x}_i = \widehat{b}_i = b_i - a_{ie}\widehat{b}_e. \qquad \square$$

Formalizing the Simplex Algorithm: Questions

Questions:

- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Formalizing the Simplex Algorithm: Questions

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- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Example before was a particularly nice one!

The formal procedure SIMPLEX

```
SIMPLEX(A, b, c)
(N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length m
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
        if a_ie > 0
            Δ_i = b_i/a_ie
        else Δ_i = ∞
    choose an index l ∈ B that minimizes Δ_i
    if Δ_l == ∞
        return "unbounded"
    else (N, B, A, b, c, ν) = PIVOT(N, B, A, b, c, ν, l, e)
for i = 1 to n
    if i ∈ B
        x̄_i = b_i
    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

The formal procedure SIMPLEX

SIMPLEX(A, b, c)

$(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length n
while some index $j \in N$ has $c_j > 0$
    choose an index $e \in N$ for which $c_e > 0$
    for each index $i \in B$
        if $a_{ie} > 0$
            $\Delta_i = b_i / a_{ie}$
        else $\Delta_i = \infty$
    choose an index $l \in B$ that minimizes Δ_i
    if $\Delta_l == \infty$
        return "unbounded"
    else $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
for $i = 1$ **to** n
    if $i \in B$
        $\bar{x}_i = b_i$
    else $\bar{x}_i = 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Returns a slack form with a feasible basic solution (if it exists)

The formal procedure SIMPLEX

```
SIMPLEX(A, b, c)
(N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length n
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
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for i = 1 to n
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    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

The formal procedure SIMPLEX

SIMPLEX(A, b, c)

1 $(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX(A, b, c)
2 let Δ be a new vector of length n
3 **while** some index $j \in N$ has $c_j > 0$
4   choose an index $e \in N$ for which $c_e > 0$
5   **for** each index $i \in B$
6   **if** $a_{ie} > 0$
7             $\Delta_i = b_i / a_{ie}$
8   **else** $\Delta_i = \infty$
9   choose an index $l \in B$ that minimizes Δ_i
10   **if** $\Delta_l == \infty$
11   **return** "unbounded"
12   **else** $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
13 **for** $i = 1$ **to** n
14   **if** $i \in B$
15         $\bar{x}_i = b_i$
16   **else** $\bar{x}_i = 0$
17 **return** $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

- terminates if all coefficients in objective function are negative
- Line 4 picks enterring variable x_e with negative coefficient
- Lines 6 – 9 pick the tightest constraint, associated with x_l
- Line 11 returns "unbounded" if there are no constraints
- Line 12 calls PIVOT, switching roles of x_l and x_e

The formal procedure SIMPLEX

SIMPLEX(A, b, c)

1 $(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX(A, b, c)
2 let Δ be a new vector of length n
3 **while** some index $j \in N$ has $c_j > 0$
4   choose an index $e \in N$ for which $c_e > 0$
5   **for** each index $i \in B$
6   **if** $a_{ie} > 0$
7             $\Delta_i = b_i / a_{ie}$
8   **else** $\Delta_i = \infty$
9   choose an index $l \in B$ that minimizes Δ_i
10   **if** $\Delta_l == \infty$
11   **return** "unbounded"
12   **else** $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
13 **for** $i = 1$ **to** n
14   **if** $i \in B$
15         $\bar{x}_i = b_i$
16   **else** $\bar{x}_i = 0$
17 **return** $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

- terminates if all coefficients in objective function are negative
- Line 4 picks enterring variable x_e with negative coefficient
- Lines $6 - 9$ pick the tightest constraint, associated with x_l
- Line 11 returns "unbounded" if there are no constraints
- Line 12 calls PIVOT, switching roles of x_l and x_e

Return corresponding solution.

The formal procedure Simplex

```
SIMPLEX(A, b, c)
1   (N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
2   let Δ be a new vector of length n
3   while some index j ∈ N has c_j > 0
4       choose an index e ∈ N for which c_e > 0
5       for each index i ∈ B
6           if a_ie > 0
7               Δ_i = b_i/a_ie
8           else Δ_i = ∞
9       choose an index l ∈ B that minimizes Δ_i
10      if Δ_l == ∞
11          return "unbounded"
12      else (N, B, A, b, c, ν) = PIVOT(N, B, A, b, c, ν, l, e)
13  for i = 1 to n
14      if i ∈ B
15          x̄_i = b_i
16      else x̄_i = 0
17  return (x̄_1, x̄_2, ..., x̄_n)
```

Returns a slack form with a feasible basic solution (if it exists)

Lemma 29.2

Suppose the call to Initialize-Simplex in line 1 returns a slack form for which the basic solution is feasible. Then if Simplex returns a solution, it is a feasible solution. If Simplex returns "unbounded", the linear program is unbounded.

The formal procedure SIMPLEX

```
SIMPLEX(A, b, c)
(N, B, A, b, c, v) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length n
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
        if a_ie > 0
            Δ_i = b_i / a_ie
        else Δ_i = ∞
    choose an index l ∈ B that minimizes Δ_i
    if Δ_l == ∞
        return "unbounded"
```

Returns a slack form with a feasible basic solution (if it exists)

Proof is based on the following three-part loop invariant:

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

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SIMPLEX(A, b, c)
1  (N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
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3  while some index j ∈ N has c_j > 0
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7              Δ_i = b_i / a_ie
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9      choose an index l ∈ B that minimizes Δ_i
10     if Δ_l == ∞
11         return "unbounded"
```

Returns a slack form with a feasible basic solution (if it exists)

Proof is based on the following three-part loop invariant:

1. the slack form is always equivalent to the one returned by INITIALIZE-SIMPLEX,
2. for each $i \in B$, we have $b_i \geq 0$,
3. the basic solution associated with the (current) slack form is feasible.

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

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Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

The formal procedure SIMPLEX

```
SIMPLEX(A, b, c)
1  (N, B, A, b, c, v) = INITIALIZE-SIMPLEX(A, b, c)
2  let Δ be a new vector of length n
3  while some index j ∈ N has c_j > 0
4      choose an index e ∈ N for which c_e > 0
5      for each index i ∈ B
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Returns a slack form with a feasible basic solution (if it exists)

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Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

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$$
\begin{aligned}
z &= && x_1 + x_2 + x_3 \\
x_4 &= 8 &&- x_1 - x_2 \\
x_5 &= && \qquad\quad x_2 - x_3
\end{aligned}
$$

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$
\begin{array}{rcrcrcrcr}
z & = & & & x_1 & + & x_2 & + & x_3 \\
x_4 & = & 8 & - & x_1 & - & x_2 & & \\
x_5 & = & & & & & x_2 & - & x_3
\end{array}
$$

Pivot with x_1 entering and x_4 leaving

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$\begin{array}{rcrcrcrcr} z &=& & & x_1 &+& x_2 &+& x_3 \\ x_4 &=& 8 &-& x_1 &-& x_2 & & \\ x_5 &=& & & & & x_2 &-& x_3 \end{array}$$

Pivot with x_1 entering and x_4 leaving

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Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$\begin{array}{rcl} z & = & x_1 + x_2 + x_3 \\ x_4 & = & 8 - x_1 - x_2 \\ x_5 & = & x_2 - x_3 \end{array}$$

Pivot with x_1 entering and x_4 leaving

$$\begin{array}{rcl} z & = & 8 + x_3 - x_4 \\ x_1 & = & 8 - x_2 - x_4 \\ x_5 & = & x_2 - x_3 \end{array}$$

Pivot with x_3 entering and x_5 leaving

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Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$
\begin{aligned}
z &= && x_1 + x_2 + x_3 \\
x_4 &= 8 &-& x_1 - x_2 \\
x_5 &= && x_2 - x_3
\end{aligned}
$$

Pivot with x_1 entering and x_4 leaving

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\end{aligned}
$$

Pivot with x_3 entering and x_5 leaving

$$
\begin{aligned}
z &= 8 + x_2 - x_4 - x_5 \\
x_1 &= 8 - x_2 - x_4 \\
x_3 &= x_2 - x_5
\end{aligned}
$$

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$$\begin{aligned} z &= && x_1 + x_2 + x_3 \\ x_4 &= 8 &-& x_1 - x_2 \\ x_5 &= && \phantom{x_1 +{}} x_2 - x_3 \end{aligned}$$

Pivot with x_1 entering and x_4 leaving

$$\begin{aligned} z &= 8 && + x_3 - x_4 \\ x_1 &= 8 &-& x_2 - x_4 \\ x_5 &= && x_2 - x_3 \end{aligned}$$

Pivot with x_3 entering and x_5 leaving

Cycling: If additionally slack at two iterations are identical, SIMPLEX fails to terminate!

$$\begin{aligned} z &= 8 + x_2 - x_4 - x_5 \\ x_1 &= 8 - x_2 - x_4 \\ x_3 &= x_2 - x_5 \end{aligned}$$

Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

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It is theoretically possible, but very rare in practice.

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2. Random rule: Choose entering variable uniformly at random

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2. Random rule: Choose entering variable uniformly at random
3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

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Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most $\binom{n+m}{m}$ iterations.

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Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most $\binom{n+m}{m}$ iterations.

Every set B of basic variables uniquely determines a slack form, and there are at most $\binom{n+m}{m}$ unique slack forms.

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Finding an Initial Solution

Finding an Initial Solution

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

Finding an Initial Solution

$$
\begin{array}{lrcrcr}
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\text{subject to} & & & & & \\
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$\Big\downarrow$ Conversion into slack form

Finding an Initial Solution

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$

Conversion into slack form

$$\begin{array}{rcrcrcr} z & = & & & 2x_1 & - & x_2 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 \end{array}$$

Basic solution $(x_1, x_2, x_3, x_4) = (0, 0, 2, -4)$ is not feasible!

Geometric Illustration

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
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 & & x_1, x_2 & & \geq & 0
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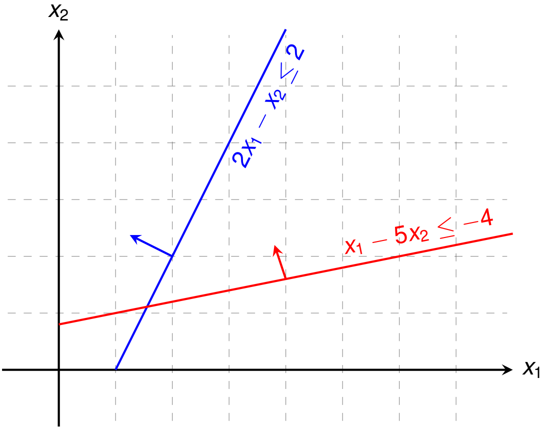

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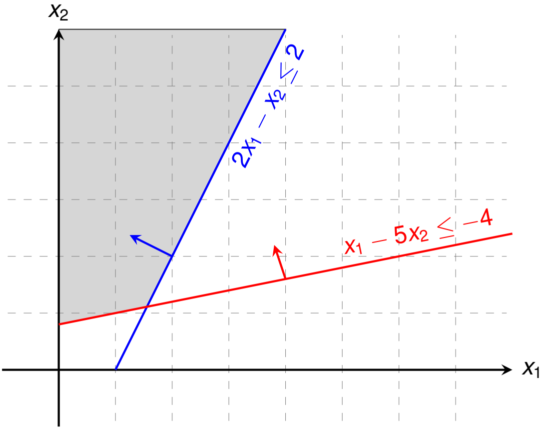
$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$



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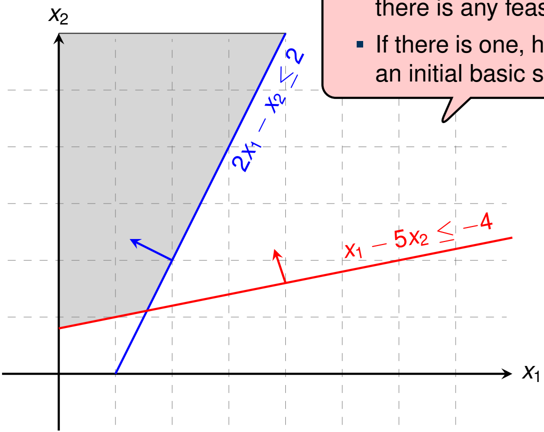

Questions:

- How to determine whether there is any feasible solution?
- If there is one, how to determine an initial basic solution?

Formulating an Auxiliary Linear Program

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad \text{for } i = 1, 2, \ldots, m, \\ x_j &\geq 0 \quad \text{for } j = 1, 2, \ldots, n \end{aligned}$$

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$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & \end{array}$$

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Lemma 29.11

Let L_{aux} be the auxiliary LP of a linear program L in standard form. Then L is feasible if and only if the optimal objective value of L_{aux} is 0.

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Formulating an Auxiliary Linear Program

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  - Since $\overline{x}_0 \geq 0$ and the objective is to maximize $-x_0$, this is optimal for L_{aux}
- "$\Leftarrow$": Suppose that the optimal objective value of L_{aux} is 0
  - Then $\overline{x}_0 = 0$, and the remaining solution values $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ satisfy L.

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$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \\ & \begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for } i = 1, 2, \ldots, m, \\ x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n \end{array} \end{array}$$

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```
INITIALIZE-SIMPLEX(A, b, c)
let k be the index of the minimum b_i
if b_k ≥ 0                      // is the initial basic solution feasible?
    return ({1, 2, ..., n}, {n + 1, n + 2, ..., n + m}, A, b, c, 0)
form L_aux by adding −x_0 to the left-hand side of each constraint
    and setting the objective function to −x_0
let (N, B, A, b, c, v) be the resulting slack form for L_aux
l = n + k
// L_aux has n + 1 nonbasic variables and m basic variables.
(N, B, A, b, c, v) = PIVOT(N, B, A, b, c, v, l, 0)
// The basic solution is now feasible for L_aux.
iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_aux is found
if the optimal solution to L_aux sets x̄_0 to 0
    if x̄_0 is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_aux, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

INITIALIZE-SIMPLEX(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
$l = n + k$
// L_{aux} has $n + 1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, \nu) = \text{PIVOT}(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for L_{aux}.
iterate the **while** loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_{aux} is found
if the optimal solution to L_{aux} sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
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Initialize-Simplex

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

Initialize-Simplex(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
  return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint
  and setting the objective function to $-x_0$
let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
$l = n + k$
// L_{aux} has $n + 1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, \nu) = \text{Pivot}(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for L_{aux}.
iterate the **while** loop of lines 3–12 of Simplex until an optimal solution
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ℓ will be the leaving variable so that x_ℓ has the most negative value.

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

INITIALIZE-SIMPLEX(A, b, c)

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if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
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$l = n + k$
// L_{aux} has $n+1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, \nu) = \text{PIVOT}(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for L_{aux}.
iterate the **while** loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_{aux} is found
if the optimal solution to L_{aux} sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_{aux}, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"

ℓ will be the leaving variable so that x_ℓ has the most negative value.

Pivot step with x_ℓ leaving and x_0 entering.

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

```
INITIALIZE-SIMPLEX(A, b, c)
let k be the index of the minimum b_i
if b_k ≥ 0                    // is the initial basic solution feasible?
    return ({1, 2, ..., n}, {n + 1, n + 2, ..., n + m}, A, b, c, 0)
form L_aux by adding −x_0 to the left-hand side of each constraint
    and setting the objective function to −x_0
let (N, B, A, b, c, v) be the resulting slack form for L_aux
l = n + k
// L_aux has n + 1 nonbasic variables and m basic variables.
(N, B, A, b, c, v) = PIVOT(N, B, A, b, c, v, l, 0)
// The basic solution is now feasible for L_aux.
iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_aux is found
if the optimal solution to L_aux sets x̄_0 to 0
    if x̄_0 is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_aux, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

ℓ will be the leaving variable so that x_ℓ has the most negative value.

Pivot step with x_ℓ leaving and x_0 entering.

This pivot step does not change the value of any variable.

Example of INITIALIZE-SIMPLEX (1/3)

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

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$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
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Formulating the auxiliary linear program

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\begin{array}{lrcrcrcr}
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\end{array}
$$

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$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{lrcrcrcr} \text{maximize} & & & & - & x_0 & & \\ \text{subject to} & & & & & & & \\ & 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\ & x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\ & & x_1, x_2, x_0 & & & & \geq & 0 \end{array}$$

Converting into slack form

Example of Initialize-Simplex (1/3)

$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 \\ \text{subject to} & \\ & \begin{array}{rcl} 2x_1 - x_2 & \leq & 2 \\ x_1 - 5x_2 & \leq & -4 \\ x_1, x_2 & \geq & 0 \end{array} \end{array}$$

$\Downarrow$ Formulating the auxiliary linear program

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & \\ & \begin{array}{rcl} 2x_1 - x_2 - x_0 & \leq & 2 \\ x_1 - 5x_2 - x_0 & \leq & -4 \\ x_1, x_2, x_0 & \geq & 0 \end{array} \end{array}$$

$\Downarrow$ Converting into slack form

$$\begin{array}{rcl} z & = & -x_0 \\ x_3 & = & 2 - 2x_1 + x_2 + x_0 \\ x_4 & = & -4 - x_1 + 5x_2 + x_0 \end{array}$$

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
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& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{lrcrcrcr}
\text{maximize} & & & & - & x_0 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
& x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
& x_1, x_2, x_0 & & & & & \geq & 0
\end{array}$$

Basic solution $(0, 0, 0, 2, -4)$ not feasible!

Converting into slack form

$$\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}$$

Example of INITIALIZE-SIMPLEX (2/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

Example of Initialize-Simplex (2/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

$\Big\downarrow$ Pivot with x_0 entering and x_4 leaving

Example of INITIALIZE-SIMPLEX (2/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

Pivot with x_0 entering and x_4 leaving

$$
\begin{array}{rcrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4
\end{array}
$$

Example of INITIALIZE-SIMPLEX (2/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

Pivot with x_0 entering and x_4 leaving

$$
\begin{array}{rcrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4
\end{array}
$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Example of Initialize-Simplex (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Optimal solution has $x_0 = 0$, hence the initial problem was feasible!

Example of INITIALIZE-SIMPLEX (3/3)

$$
\begin{aligned}
z &= && - && x_0 \\
x_2 &= \frac{4}{5} && - && \frac{x_0}{5} && + && \frac{x_1}{5} && + && \frac{x_4}{5} \\
x_3 &= \frac{14}{5} && + && \frac{4x_0}{5} && - && \frac{9x_1}{5} && + && \frac{x_4}{5}
\end{aligned}
$$

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{aligned}
z &= && - && x_0 \\
x_2 &= \frac{4}{5} && - && \frac{x_0}{5} && + && \frac{x_1}{5} && + && \frac{x_4}{5} \\
x_3 &= \frac{14}{5} && + && \frac{4x_0}{5} && - && \frac{9x_1}{5} && + && \frac{x_4}{5}
\end{aligned}$$

Set $x_0 = 0$ and express objective function by non-basic variables

Example of INITIALIZE-SIMPLEX (3/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & - & x_0 & & & & \\
x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5}
\end{array}
$$

Set $x_0 = 0$ and express objective function by non-basic variables

$2x_1 - x_2 = 2x_1 - (\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5})$

$$
\begin{array}{rcrcrcr}
z & = & -\frac{4}{5} & + & \frac{9x_1}{5} & - & \frac{x_4}{5} \\
x_2 & = & \frac{4}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5}
\end{array}
$$

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Set $x_0 = 0$ and express objective function by non-basic variables

$2x_1 - x_2 = 2x_1 - (\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5})$

$$\begin{array}{rcrcrcr} z & = & -\frac{4}{5} & + & \frac{9x_1}{5} & - & \frac{x_4}{5} \\ x_2 & = & \frac{4}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Basic solution $(0, \frac{4}{5}, \frac{14}{5}, 0)$, which is feasible!

Example of Initialize-Simplex (3/3)

$$\begin{aligned} z &= & & - x_0 & & & & \\ x_2 &= \frac{4}{5} & & - \frac{x_0}{5} & &+ \frac{x_1}{5} & &+ \frac{x_4}{5} \\ x_3 &= \frac{14}{5} & &+ \frac{4x_0}{5} & &- \frac{9x_1}{5} & &+ \frac{x_4}{5} \end{aligned}$$

$2x_1 - x_2 = 2x_1 - (\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5})$

Set $x_0 = 0$ and express objective function by non-basic variables

$$\begin{aligned} z &= -\frac{4}{5} & &+ \frac{9x_1}{5} & &- \frac{x_4}{5} \\ x_2 &= \frac{4}{5} & &+ \frac{x_1}{5} & &+ \frac{x_4}{5} \\ x_3 &= \frac{14}{5} & &- \frac{9x_1}{5} & &+ \frac{x_4}{5} \end{aligned}$$

Basic solution $(0, \frac{4}{5}, \frac{14}{5}, 0)$, which is feasible!

Lemma 29.12

If a linear program L has no feasible solution, then Initialize-Simplex returns "infeasible". Otherwise, it returns a valid slack form for which the basic solution is feasible.

Fundamental Theorem of Linear Programming

Theorem 29.13 (Fundamental Theorem of Linear Programming)

Any linear program L, given in standard form, either

1. has an optimal solution with a finite objective value,
2. is infeasible, or
3. is unbounded.

If L is infeasible, SIMPLEX returns "infeasible". If L is unbounded, SIMPLEX returns "unbounded". Otherwise, SIMPLEX returns an optimal solution with a finite objective value.

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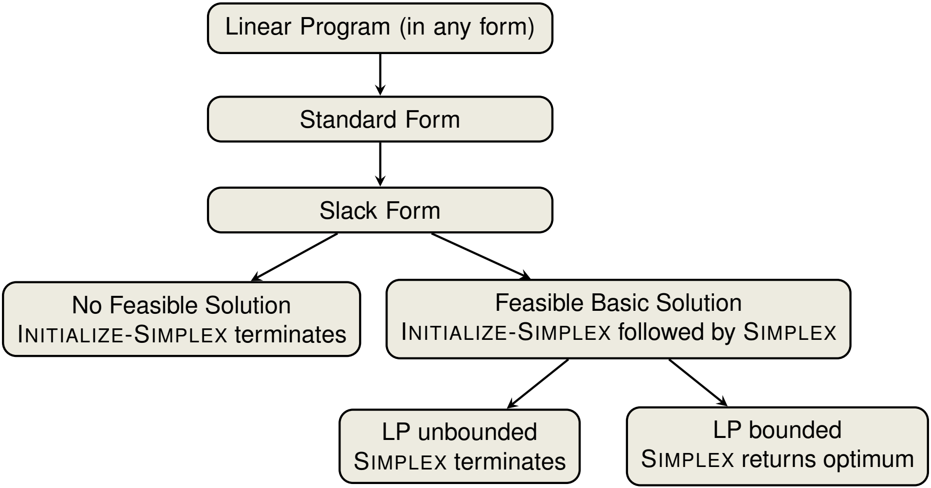
If L is infeasible, SIMPLEX returns "infeasible". If L is unbounded, SIMPLEX returns "unbounded". Otherwise, SIMPLEX returns an optimal solution with a finite objective value.

Proof requires the concept of duality, which is not covered in this course (for details see CLRS3, Chapter 29.4)

Workflow for Solving Linear Programs



Linear Programming and Simplex: Summary and Outlook

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- extremely versatile tool for modelling problems of all kinds

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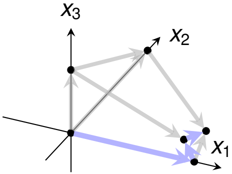
- In practice: usually terminates in polynomial time, i.e., $O(m+n)$



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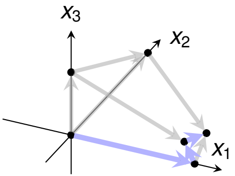

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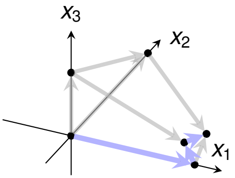
Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?



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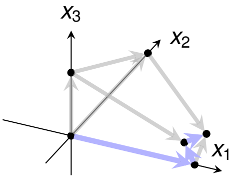

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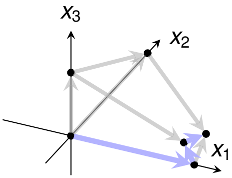
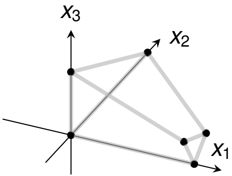


Polynomial-Time Algorithms

- Interior-Point Methods: traverses the interior of the feasible set of solutions (not just vertices!)



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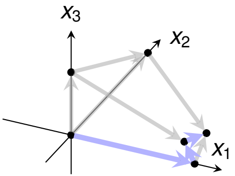

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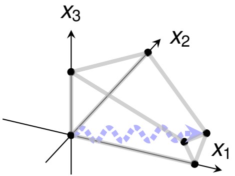

IV. Approximation Algorithms: Covering Problems

Thomas Sauerwald

Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Vertex Cover

The Set-Covering Problem

Motivation

Many fundamental problems are **NP-complete**, yet they are too important to be abandoned.

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Examples: HAMILTON, 3-SAT, VERTEX-COVER, KNAPSACK,...

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Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

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1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

We will call these **approximation algorithms**.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

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An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

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- Maximization problem: $\frac{C^*}{C} \geq 1$
- Minimization problem: $\frac{C}{C^*} \geq 1$

This covers both maximization and minimization problems.

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For many problems: tradeoff between runtime and approximation ratio.

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Outline

Introduction

Vertex Cover

The Set-Covering Problem

The Vertex-Cover Problem

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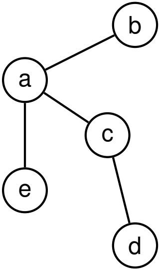
- Goal: Find a minimum-cardinality subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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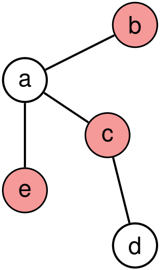

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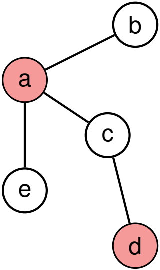

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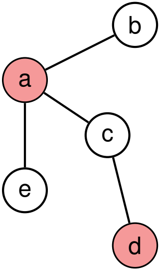

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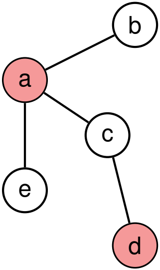

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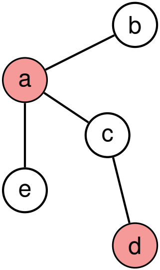

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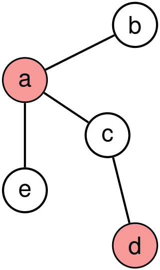

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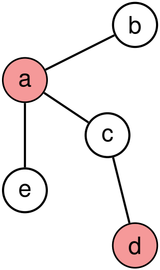

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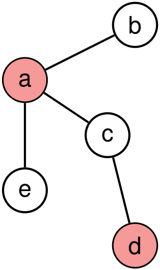

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Perform all tasks with the minimal amount of resources
- Extensions: weighted vertices or hypergraphs ($\rightsquigarrow$ Set-Covering Problem)

An Approximation Algorithm based on Greedy

APPROX-VERTEX-COVER(G)

```
C = ∅
E' = G.E
while E' ≠ ∅
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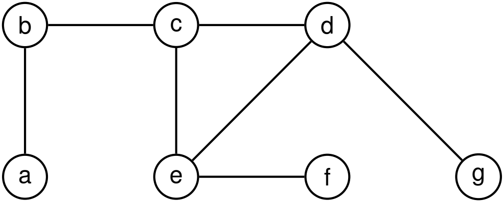

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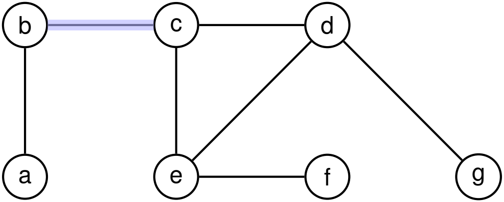

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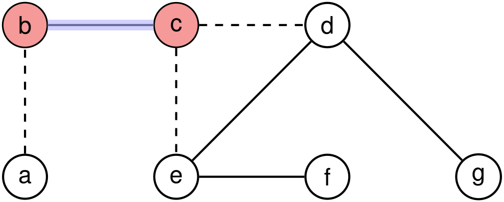

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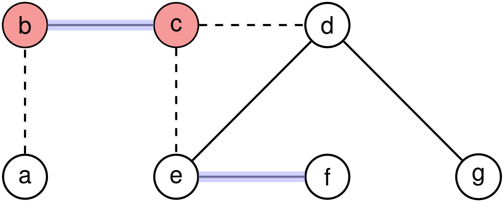

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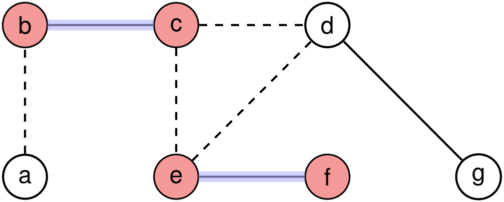

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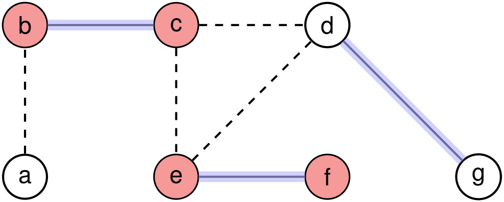

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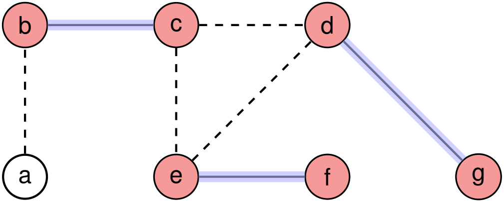

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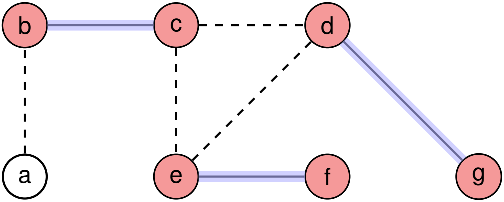

APPROX-VERTEX-COVER produces a set of size 6.

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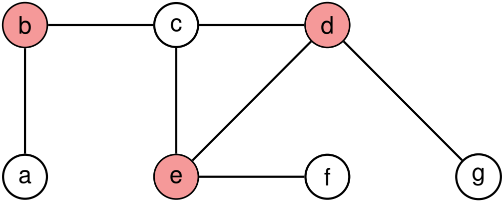

The optimal solution has size 3.

Analysis of Greedy for Vertex Cover

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APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

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A "vertex-based" Greedy that adds **one** vertex at each iteration fails to achieve an approximation ratio of 2 (Exercise)!

We can bound the size of the returned solution without knowing the (size of an) optimal solution!

Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof:

- Running time is $O(V + E)$ (using adjacency lists to represent E')
- Let $A \subseteq E$ denote the set of edges picked in line 4
- Every optimal cover C^* must include at least one endpoint of edges in A, and edges in A do not share a common endpoint: $\boxed{|C^*| \geq |A|}$
- Every edge in A contributes 2 vertices to $|C|$: $\boxed{|C| = 2|A| \leq 2|C^*|.}$ $\square$

Solving Special Cases

Strategies to cope with NP-complete problems

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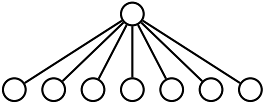

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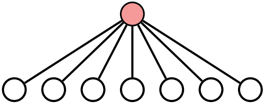

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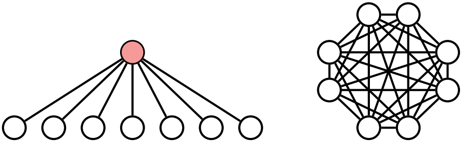

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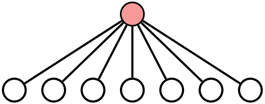

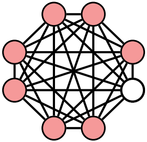

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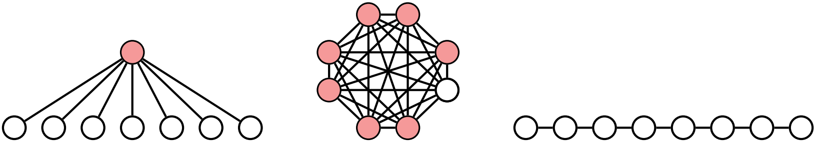

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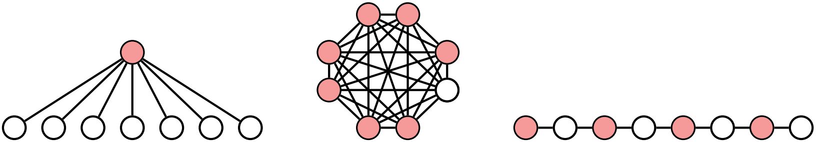
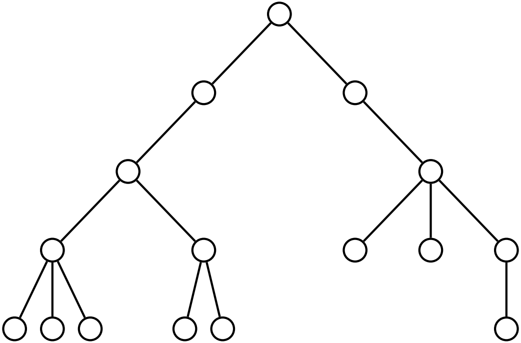


Vertex Cover on Trees



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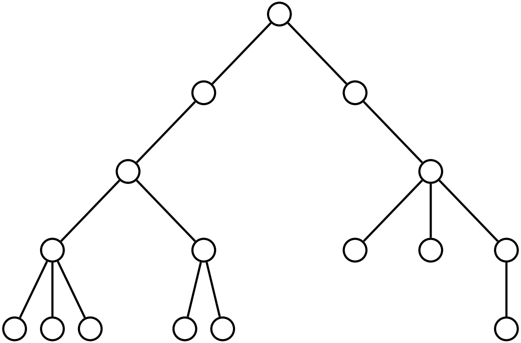

There exists an optimal vertex cover which does not include any leaves.

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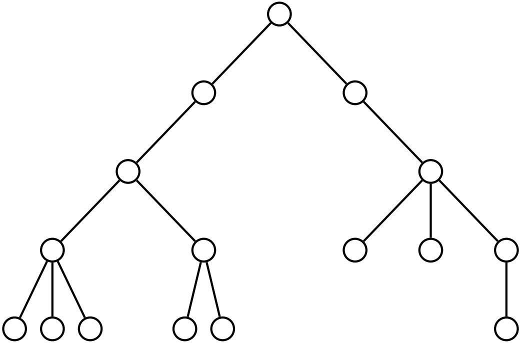

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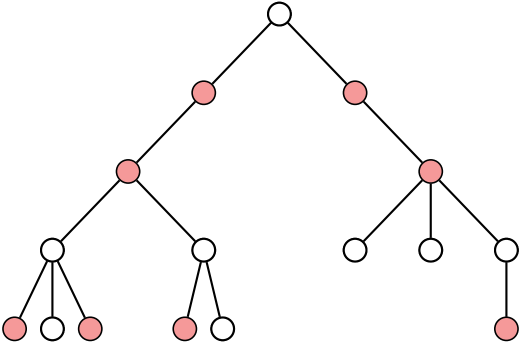

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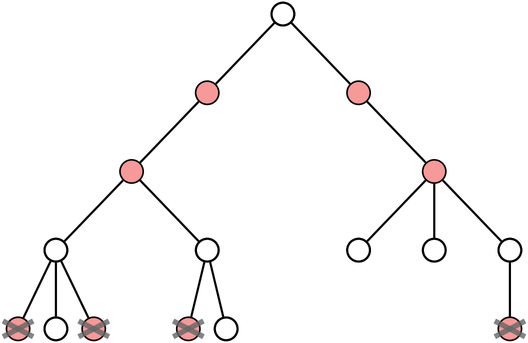

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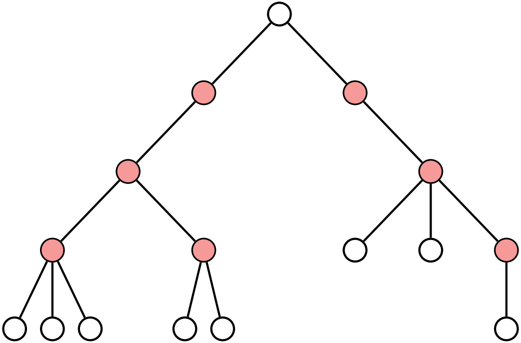

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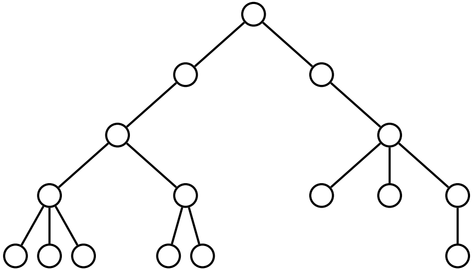
Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example



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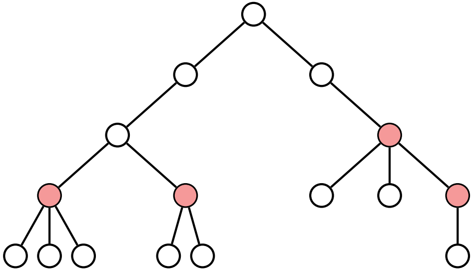

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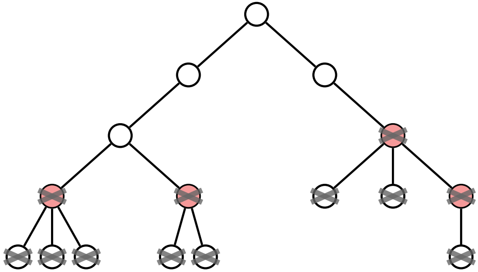

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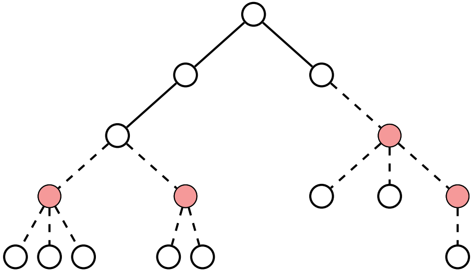

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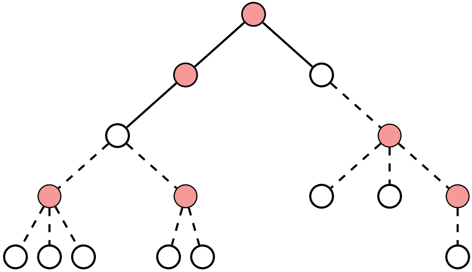

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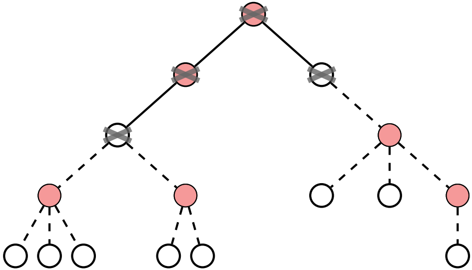

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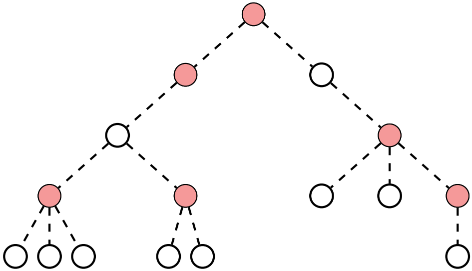

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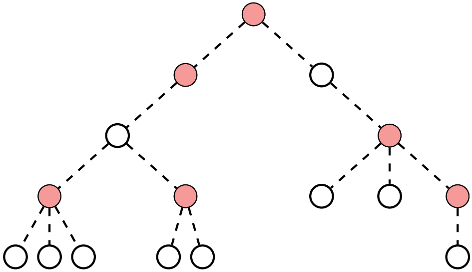

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Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory
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Focus on instances where the minimum vertex cover is small, that is, **less or equal** than some given integer k.

Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

Consider a graph $G = (V, E)$, edge $\{u, v\} \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

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Reminiscent of Dynamic Programming.

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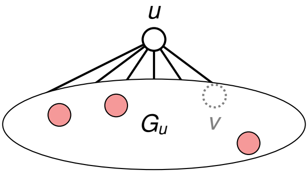

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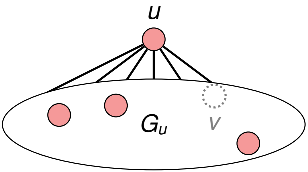

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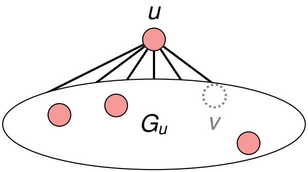

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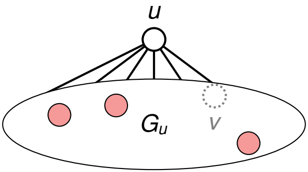
Removing u from C yields a vertex cover of G_u which is of size $k-1$. $\square$



A More Efficient Search Algorithm

```
VERTEX-COVER-SEARCH(G, k)
If E = ∅ return ∅
If k = 0 and E ≠ ∅ return ⊥
Pick an arbitrary edge (u, v) ∈ E
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if S_1 ≠ ⊥ return S_1 ∪ {u}
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Correctness follows by the Substructure Lemma and induction.

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exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)

Outline

Introduction

Vertex Cover

The Set-Covering Problem

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Set Cover Problem

- Given: set X of size n and family of subsets $\mathcal{F}$
- Goal: Find a minimum-size subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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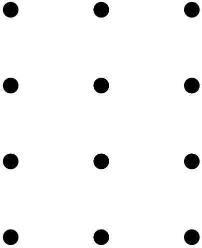

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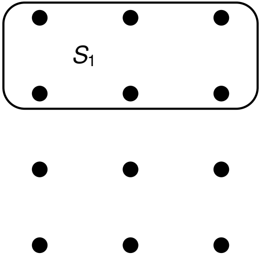

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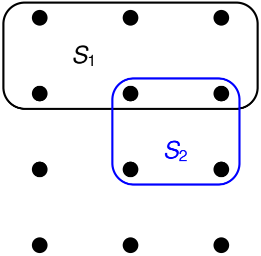

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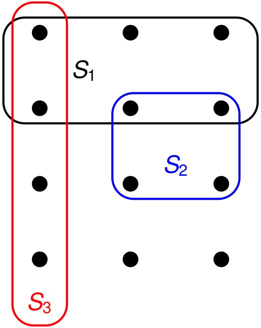

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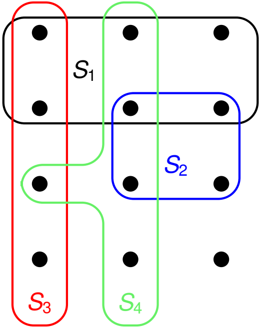

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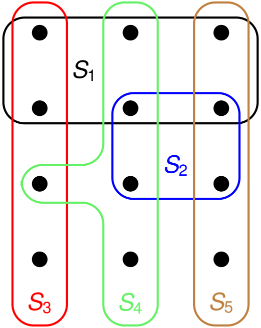

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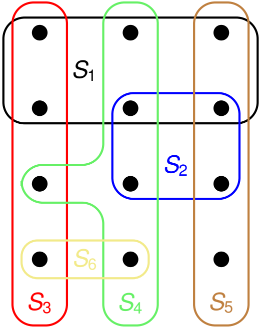

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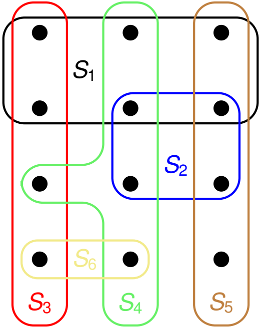

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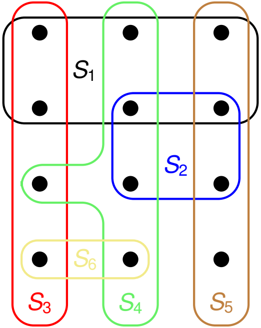

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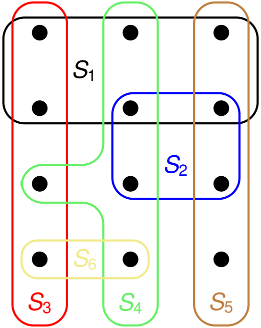

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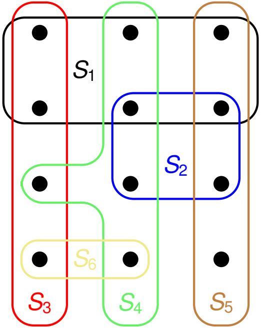

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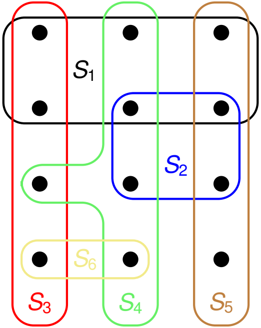

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- generalisation of the vertex-cover problem and hence also NP-hard.
- models resource allocation problems

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Strategy: Pick the set S that covers the largest number of uncovered elements.

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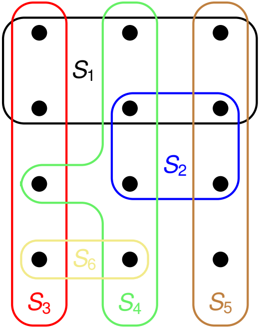

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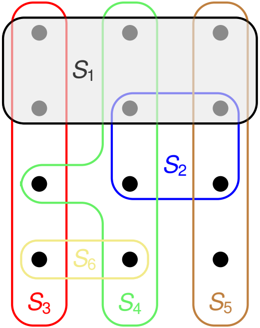

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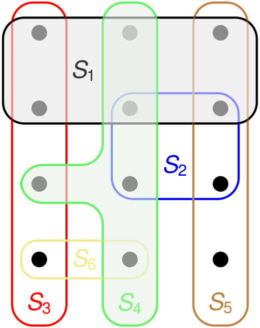

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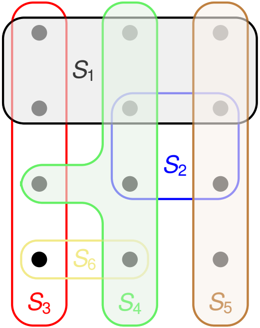

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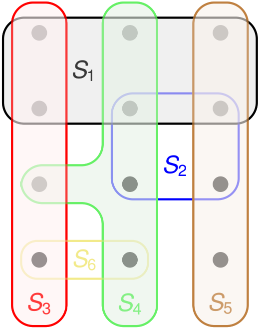

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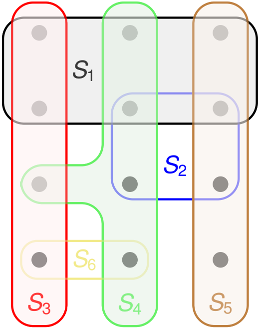

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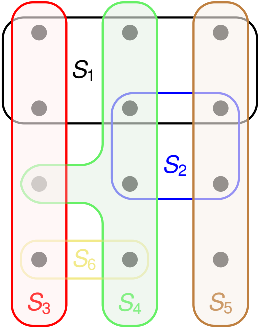

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Optimal cover is $\mathcal{C} = \{S_3, S_4, S_5\}$

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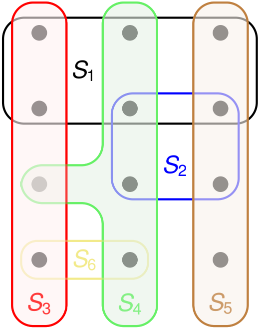
Can be easily implemented to run in time polynomial in $|X|$ and $|\mathcal{F}|$



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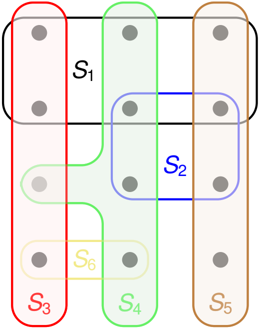

How good is the approximation ratio?

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Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

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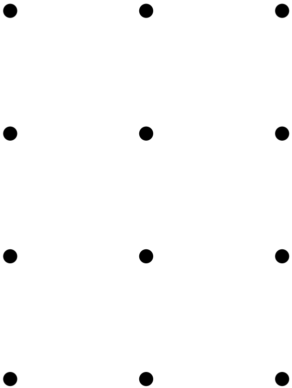
Notice that in the mathematical analysis, S_i is the set chosen in iteration i - not to be confused with the sets $S_1, S_2, \ldots, S_6$ in the example.

Illustration of Costs for Greedy picking S_1, S_4, S_5 and S_3



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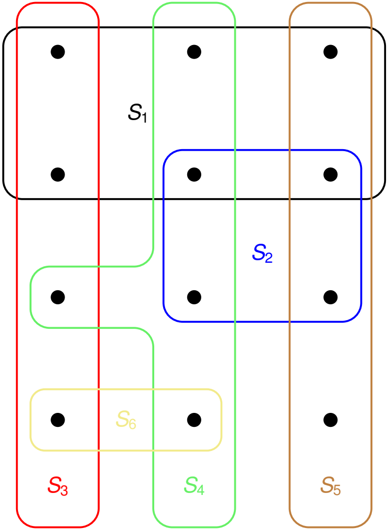

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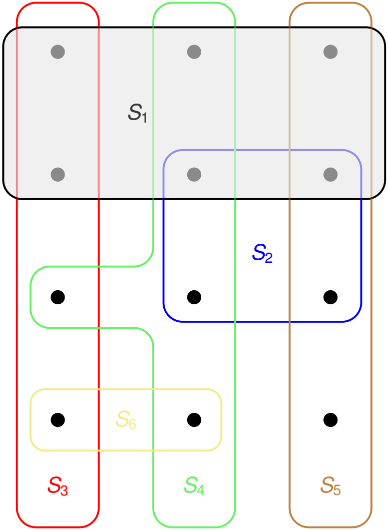

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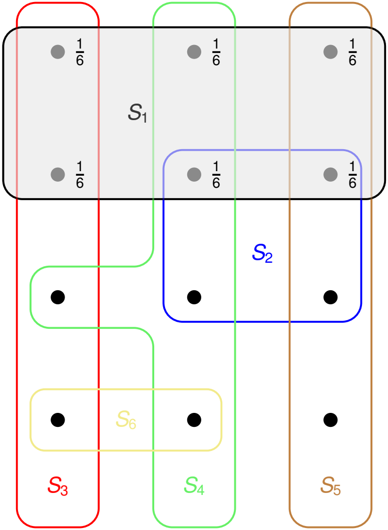

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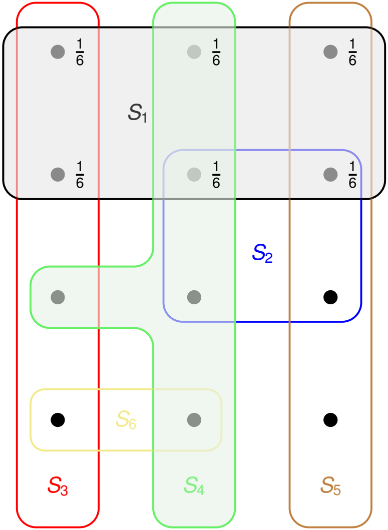

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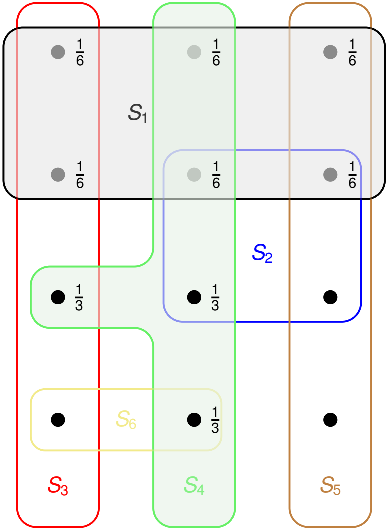

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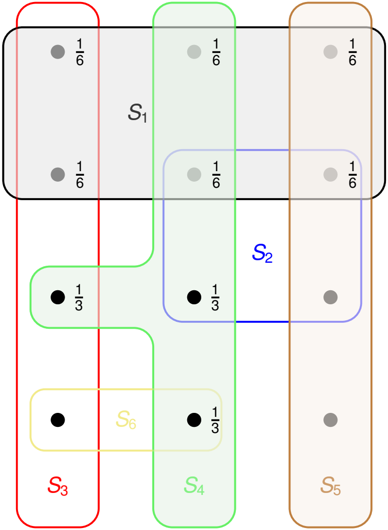

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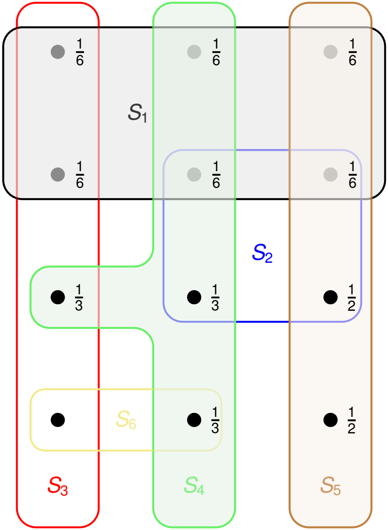

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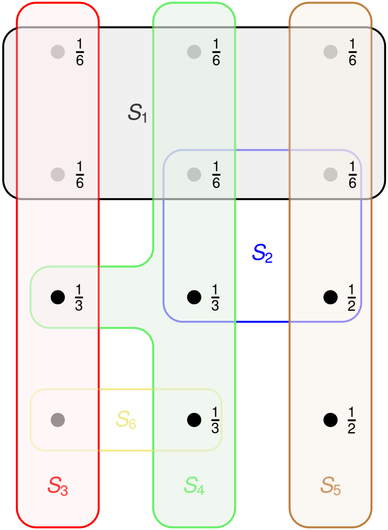

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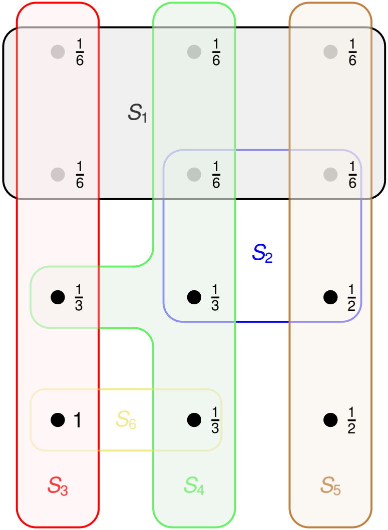

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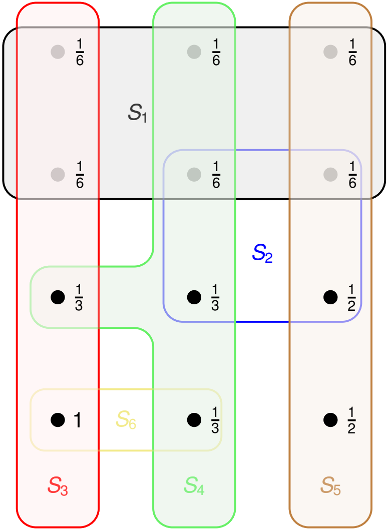

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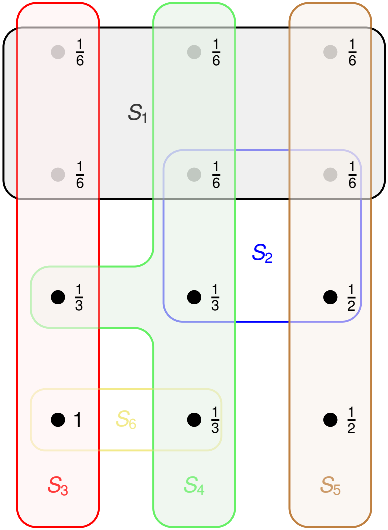

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Key Inequality: $\sum_{x \in S} c_x \leq H(|S|)$.

Proof of Theorem 35.4 (1/2)

Definition of cost

If x is covered for the first time by a set S_i, then $c_x := \frac{1}{\left|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})\right|}$.

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Remaining uncovered elements in S

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Set-Covering Problem (Summary)

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S| \colon |S| \in \mathcal{F}\}) \leq \ln(n) + 1.$$

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The same approach also gives an approximation ratio of $O(\ln(n))$ if there exists a cost function $c : S \to \mathbb{Z}^+$

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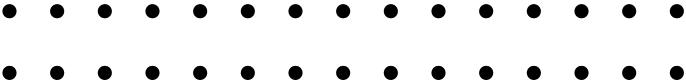

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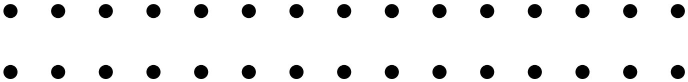

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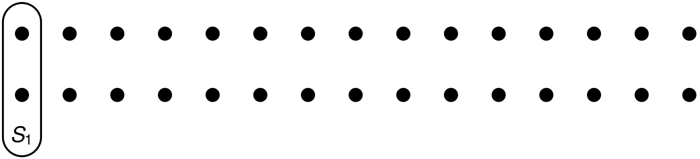

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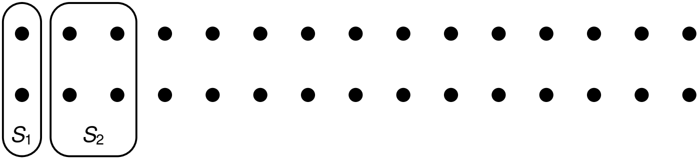

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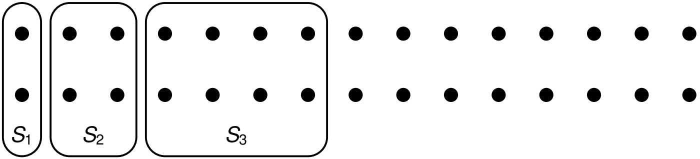

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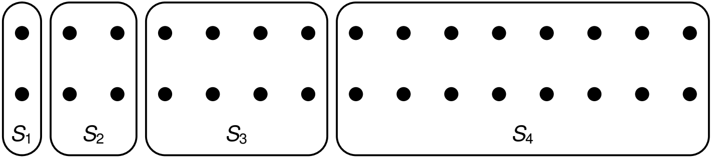

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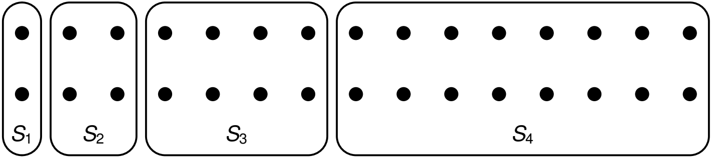

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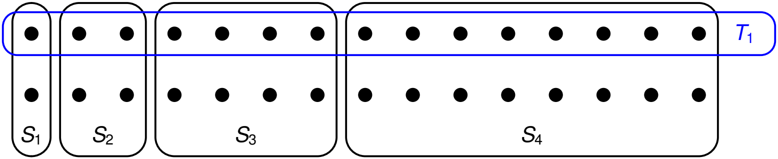

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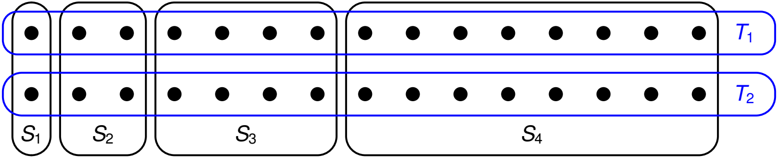

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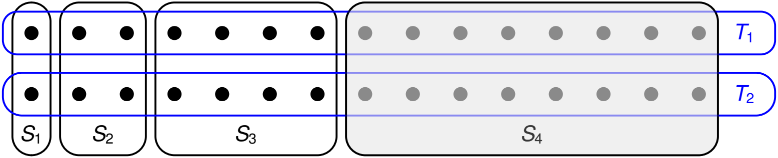

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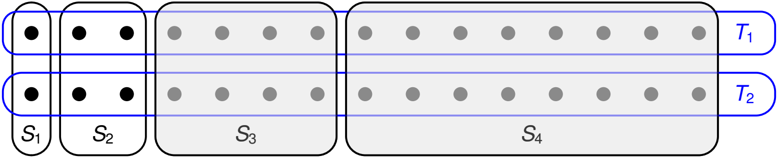

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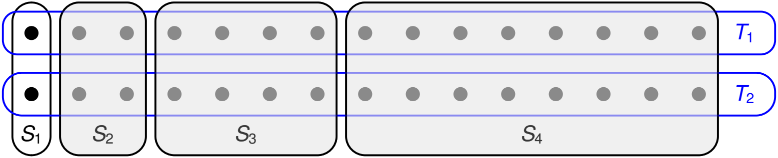

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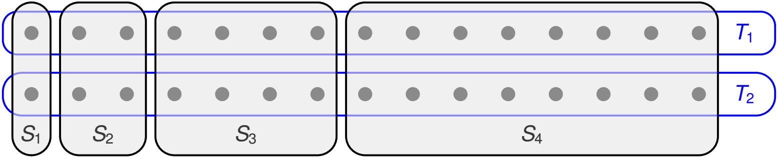

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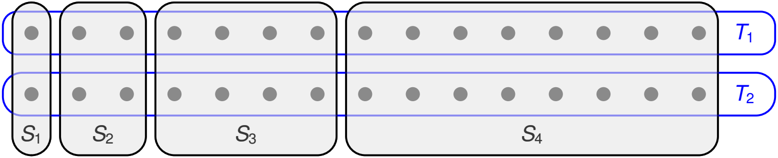

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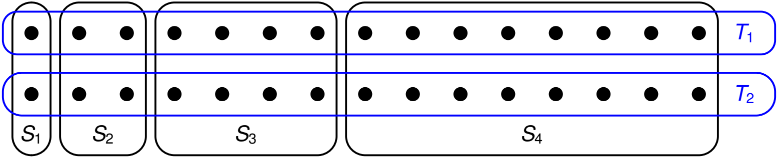

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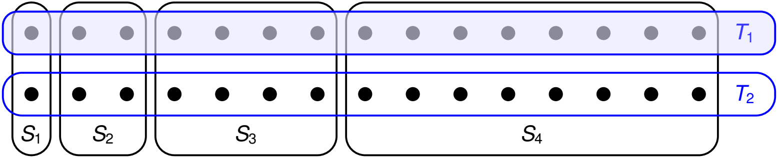

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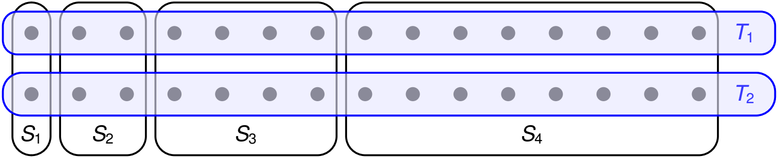

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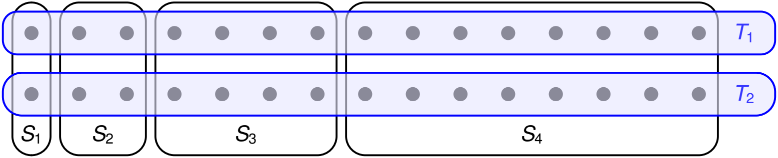

Solution of Greedy consists of k sets.

Optimum consists of 2 sets.

V. Approximation Algorithms via Exact Algorithms

Thomas Sauerwald

Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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This problem is NP-hard

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$x_1 = 10$

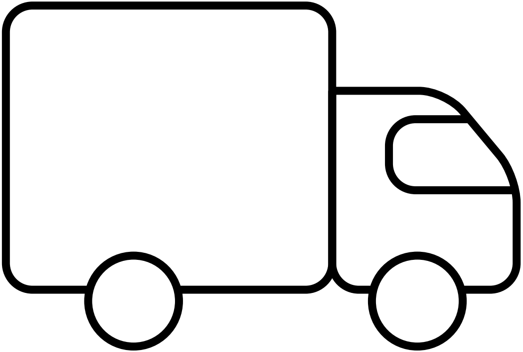
$x_2 = 4$

$x_3 = 5$

$x_4 = 6$

$x_5 = 1$

$t = 13$ tons



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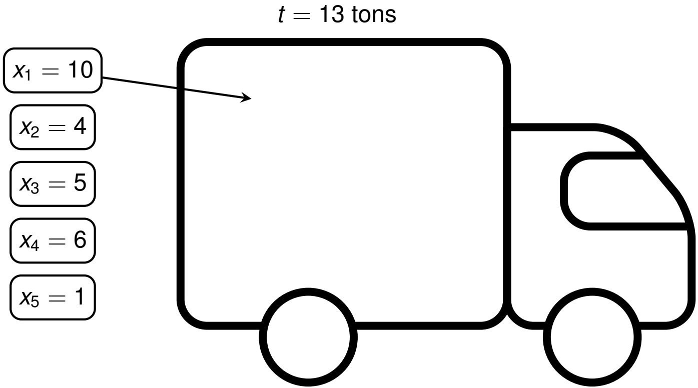

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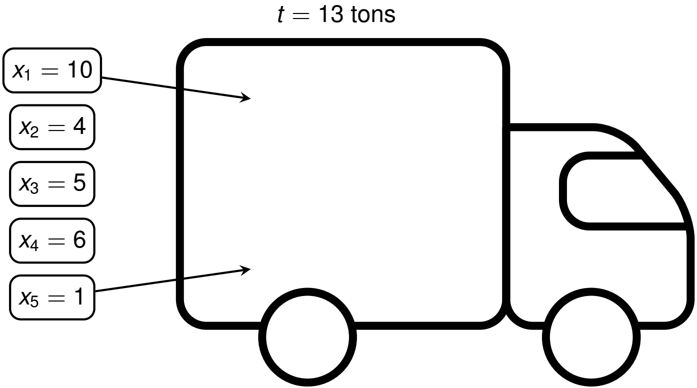

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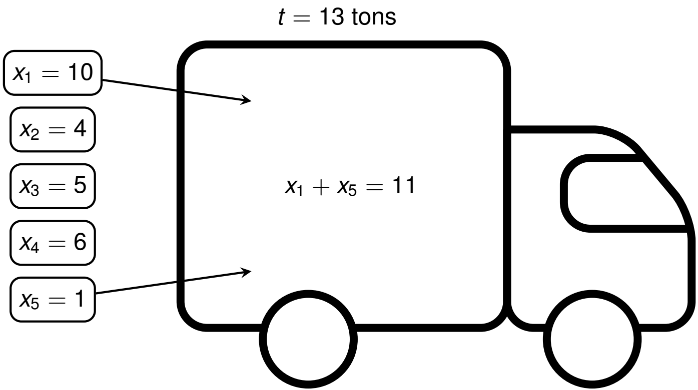

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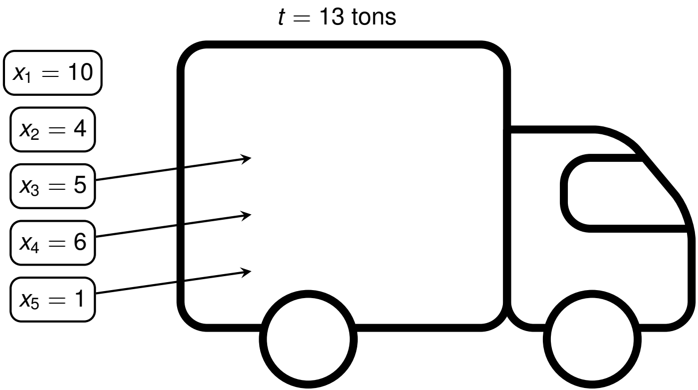

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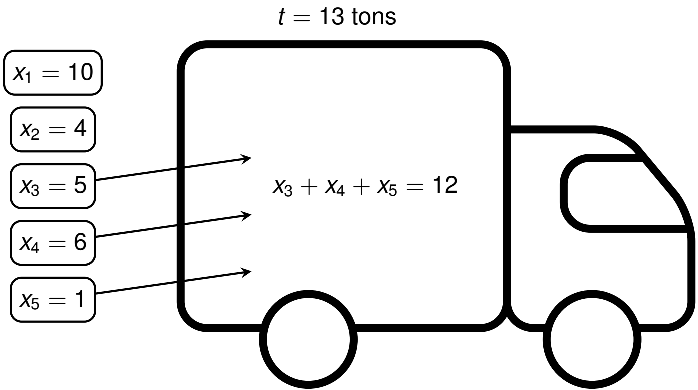

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

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EXACT-SUBSET-SUM(S, t)

$n = |S|$
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Returns the merged list (in sorted order and without duplicates)

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implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

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can be shown by induction on n

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Better runtime if t and/or $|L_i|$ are small.

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Idea: Don't need to maintain two values in L which are close to each other.

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- Given a trimming parameter $0 < \delta < 1$

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Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

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Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

- $L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

Towards a FPTAS

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Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

```
TRIM(L, δ)
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
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        append y_i onto the end of L'
        last = y_i
return L'
```

TRIM works in time $\Theta(m)$, if L is given in sorted order.

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
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for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
    if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
        append y_i onto the end of L'
        $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
last $= y_1$
for $i = 2$ **to** m
    if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
        append y_i onto the end of L'
        last $= y_i$
return L'

$\delta = 0.1$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

```
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
last $= y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    last $= y_i$
return L'

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
last $= y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    last $= y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
last $= y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    last $= y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
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for $i = 2$ **to** m
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    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, \overset{\overset{\text{last}}{\downarrow}}{23}, \underset{\underset{\text{i}}{\uparrow}}{24}, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
    if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
        append y_i onto the end of L'
        $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

Illustration of the Trim Operation

$\textsc{Trim}(L, \delta)$

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
    if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
        append y_i onto the end of L'
        $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

The FPTAS

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    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

The FPTAS

$\text{APPROX-SUBSET-SUM}(S, t, \epsilon)$

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

$\text{EXACT-SUBSET-SUM}(S, t)$

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
  $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
  $\boxed{L_i = \text{TRIM}(L_i, \epsilon/2n)}$
  remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
  $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
  remove from L_i every element that is greater than t
return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

EXACT-SUBSET-SUM(S, t)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

Repeated application of TRIM to make sure L_i's remain short.

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Solution is a careful choice of δ!

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i =$ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i =$ TRIM$(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      L_i = TRIM(L_i, ε/2n)
      remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101\rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
5 $\quad L_i = \text{TRIM}(L_i, \epsilon/2n)$
6 $\quad$ remove from L_i every element that is greater than t
7 let z^* be the largest value in L_n
8 **return** z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
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3 **for** $i = 1$ **to** n
4 $\quad L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
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- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$

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- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

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```
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- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$

Running through an Example

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```
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- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
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- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

Running through an Example

```
Approx-Subset-Sum(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = Merge-Lists(L_{i-1}, L_{i-1} + x_i)
5      L_i = Trim(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
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- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
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Running through an Example

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```
1  n = |S|
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- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

Running through an Example

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```
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- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404\rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404\rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302\rangle$

Returned solution $z^* = 302$, which is 2% within the optimum $307 = 104 + 102 + 101$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

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$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y$$

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$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \stackrel{y=y^*,i=n}{\Rightarrow}$$

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$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n,$$

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Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1 + \frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1 + \frac{\epsilon/2}{n}\right)^n \overset{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

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Taylor approximation of e

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$$\leq 1 + \epsilon/2 + (\epsilon/2)^2$$

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Taylor approximation of e

$$\leq 1 + \epsilon/2 + (\epsilon/2)^2 \leq 1 + \epsilon$$

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$$\log_{1+\epsilon/(2n)} t + 2 =$$

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$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1+\epsilon/(2n))} + 2$$

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$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2$$

For $x > -1$, $\ln(1 + x) \geq \frac{x}{1+x}$

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$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2$$

$$\leq \frac{2n(1 + \epsilon/(2n)) \ln t}{\epsilon} + 2$$

For $x > -1$, $\ln(1 + x) \geq \frac{x}{1+x}$

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$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\begin{aligned} \log_{1+\epsilon/(2n)} t + 2 &= \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2 \\ &\leq \frac{2n(1 + \epsilon/(2n)) \ln t}{\epsilon} + 2 \\ &< \frac{3n \ln t}{\epsilon} + 2. \end{aligned}$$

For $x > -1$, $\ln(1 + x) \geq \frac{x}{1+x}$

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For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

- This bound on $|L_i|$ is polynomial in the size of the input and in $1/\epsilon$.

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

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Need $\log(t)$ bits to represent t and n bits to represent S

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t

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Algorithm very similar to APPROX-SUBSET-SUM

Theorem

There is a FPTAS for the Knapsack problem.

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$

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- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
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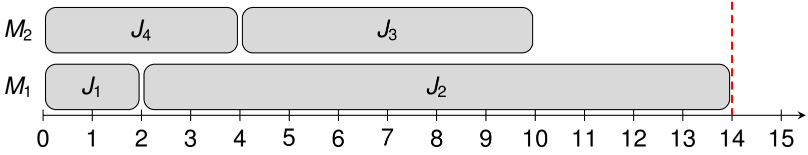

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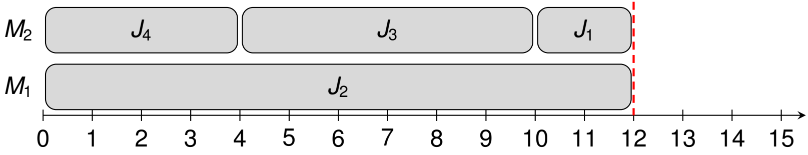

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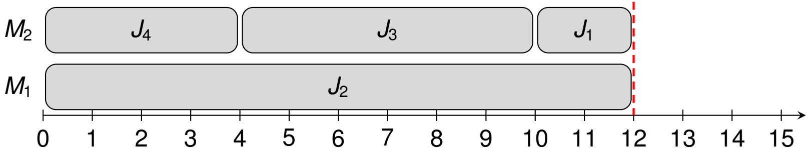
For the analysis, it will be convenient to denote by C_i the completion time of a machine i.



NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

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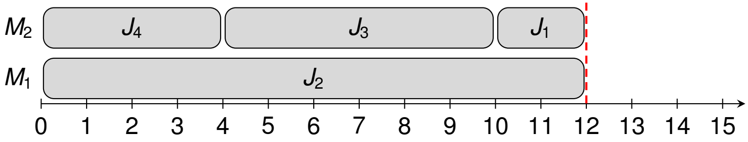

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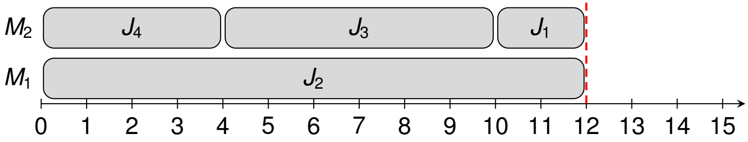

LIST SCHEDULING$(J_1, J_2, \ldots, J_n, m)$

while there exists an unassigned job

    Schedule job on the machine with the least load

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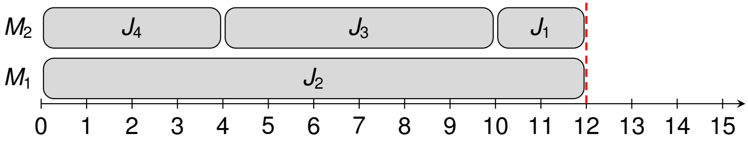

Equivalent to the following Online Algorithm [CLRS]:
Whenever a machine is idle, schedule any job that has not yet been scheduled.

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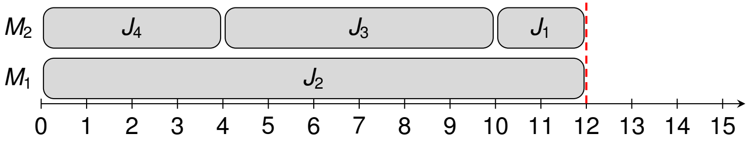

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How good is this most basic Greedy Approach?

List Scheduling Analysis (Observations)

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Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C_{\max}^{*} \geq \max_{1 \leq k \leq n} p_k .$$

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b. The total processing times of all n jobs equals $\sum_{k=1}^{n} p_k$

$\Rightarrow$ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ $\square$

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

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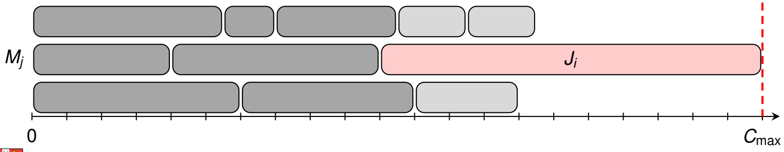

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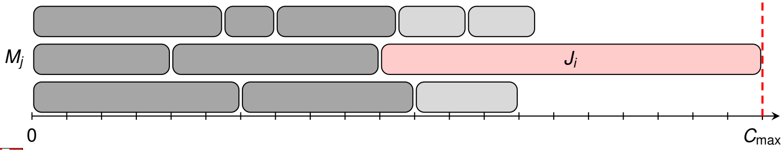

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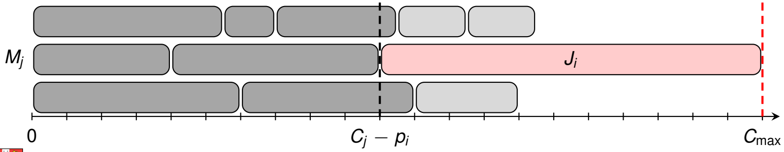

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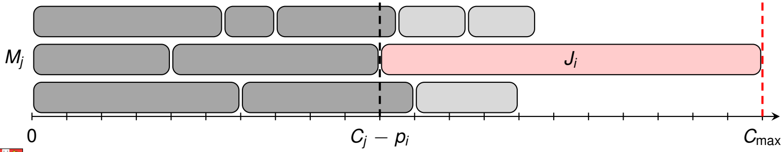

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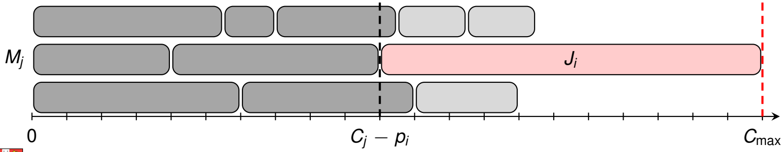

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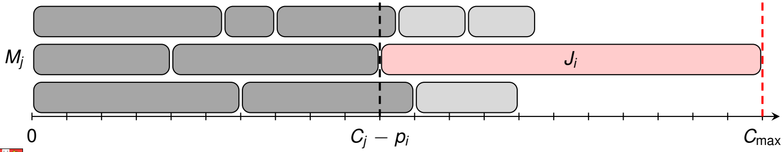

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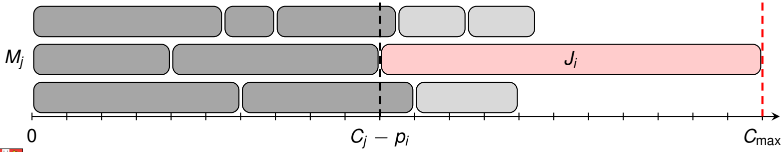

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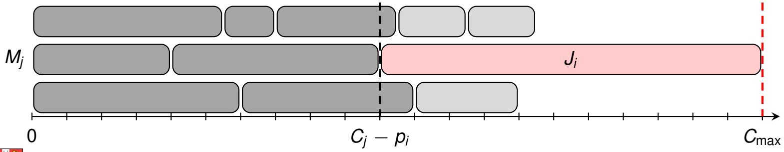

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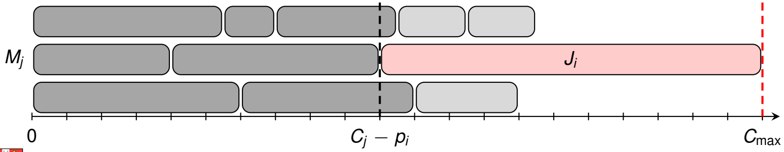

Improving Greedy

Analysis can be shown to be almost tight. Is there a better algorithm?

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The problem of the List-Scheduling Approach were the large jobs

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LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)

1: Sort jobs decreasingly in their processing times
2: **for** $i = 1$ to m
3: $\quad C_i = 0$
4: $\quad S_i = \emptyset$
5: **end for**
6: **for** $j = 1$ to n
7: $\quad i = \mathrm{argmin}_{1 \le k \le m} C_k$
8: $\quad S_i = S_i \cup \{j\}, C_i = C_i + p_j$
9: **end for**
10: **return** $S_1, \ldots, S_m$

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

Analysis can be shown to be almost tight. Is there a better algorithm?

LEAST PROCESSING TIME$(J_1, J_2, \ldots, J_n, m)$

```
Sort jobs decreasingly in their processing times
for i = 1 to m
    C_i = 0
    S_i = ∅
end for
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    S_i = S_i ∪ {j}, C_i = C_i + p_j
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Runtime:

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Runtime:

- $O(n \log n)$ for sorting
- $O(n \log m)$ for extracting (and re-inserting) the minimum (use priority queue).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

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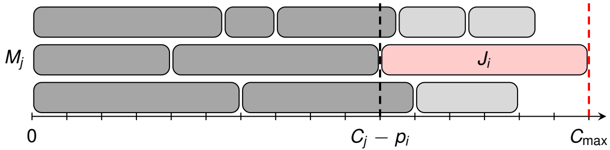

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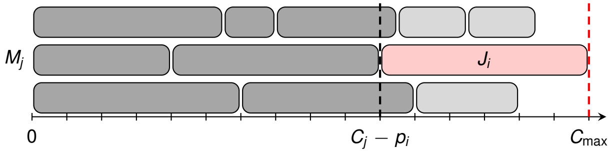

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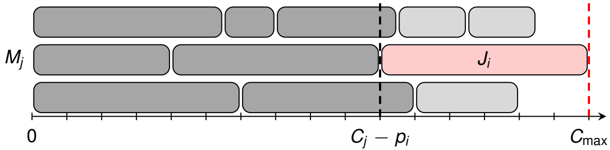
$$C_{\max} = C_j = (C_j - p_i) + p_i \leq C^*_{\max} + \frac{1}{2} C^*_{\max}$$

This is for the case $i \geq m+1$ (otherwise, an even stronger inequality holds)



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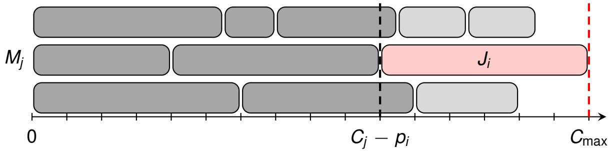

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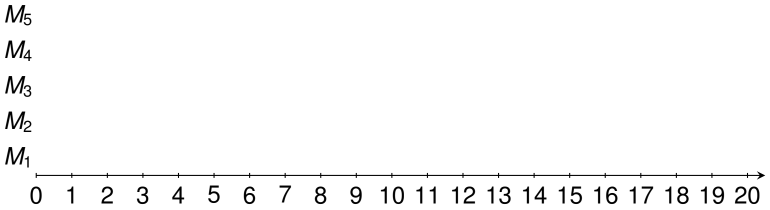

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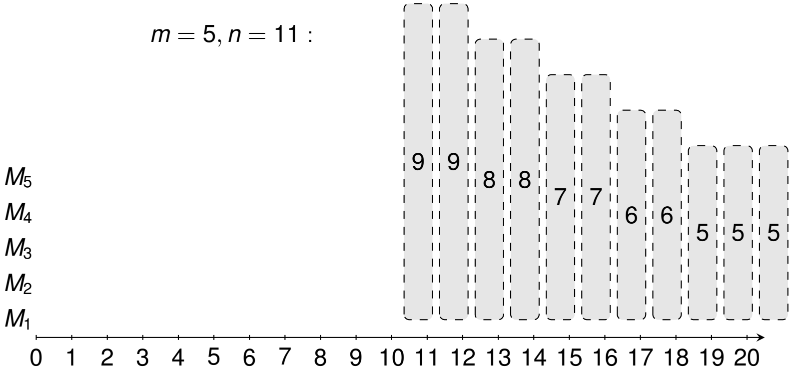

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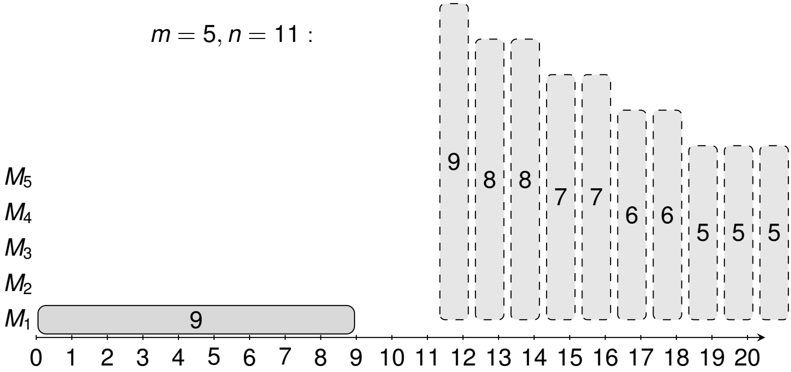

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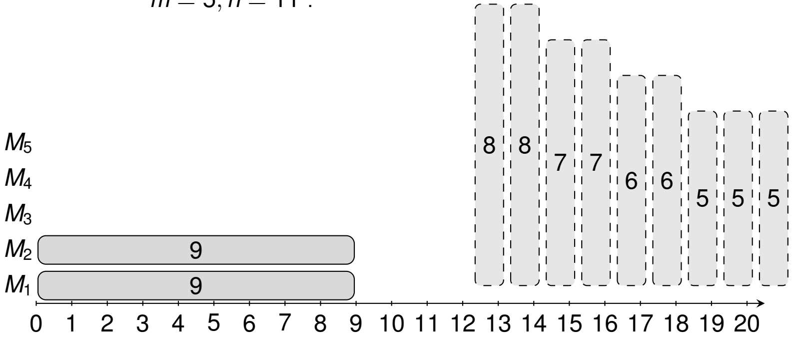

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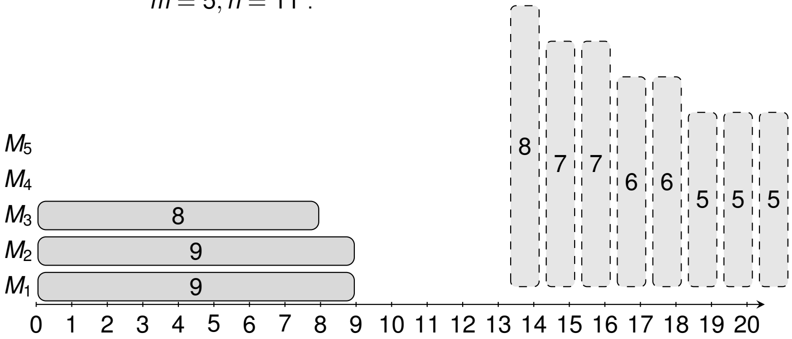

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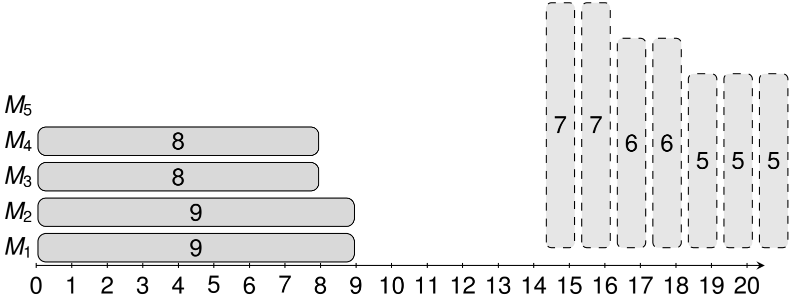

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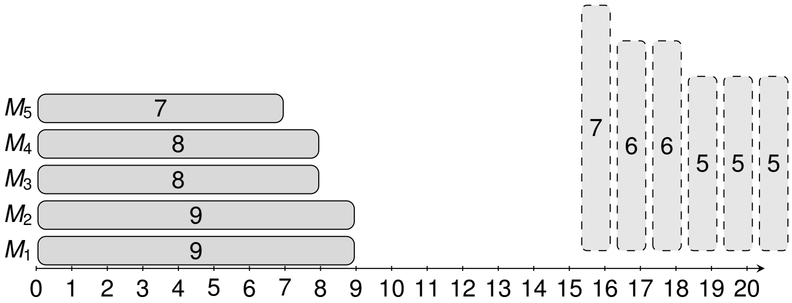

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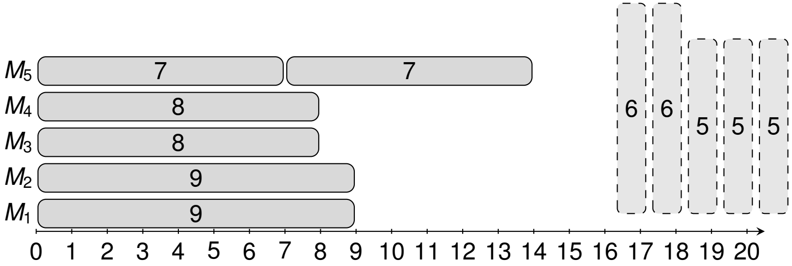

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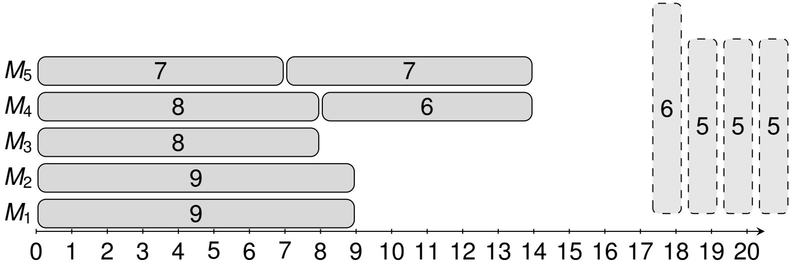

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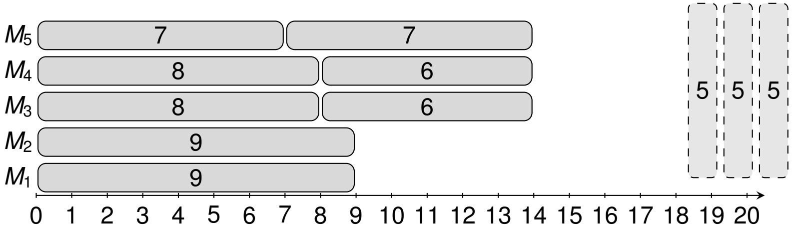

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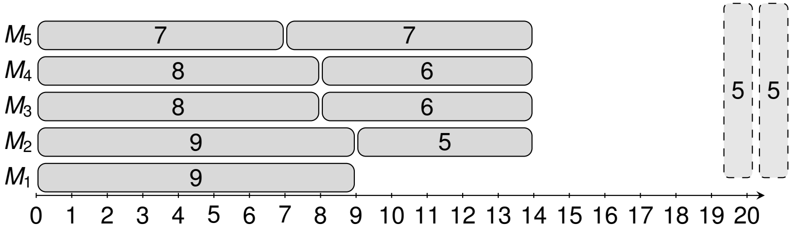

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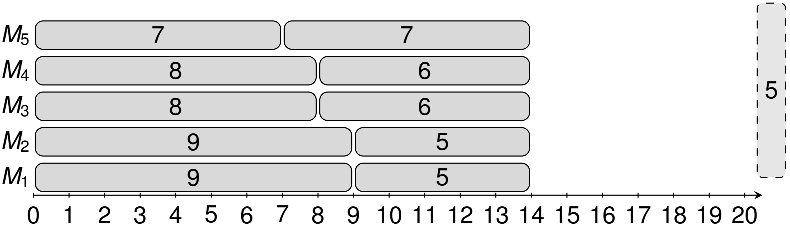

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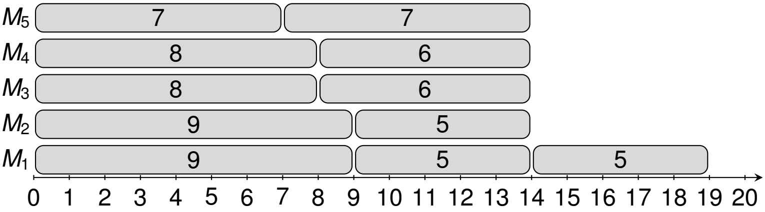

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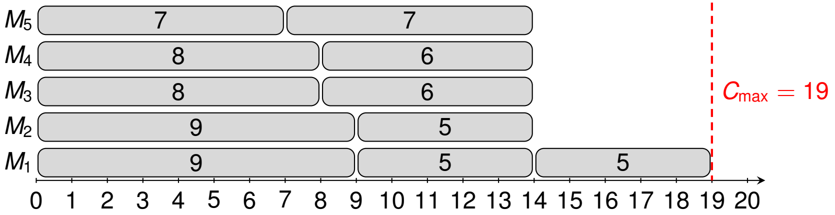

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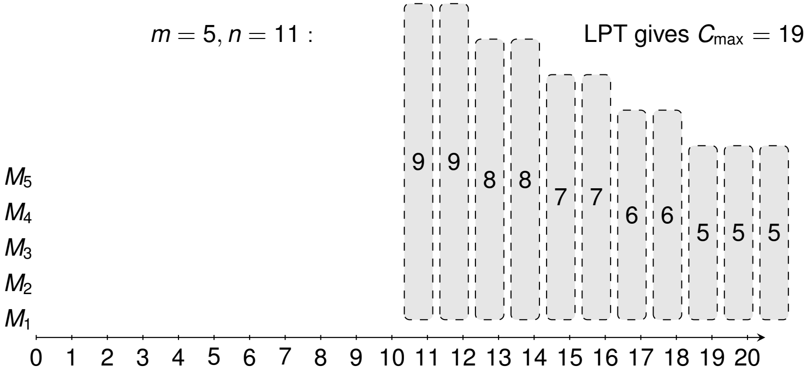

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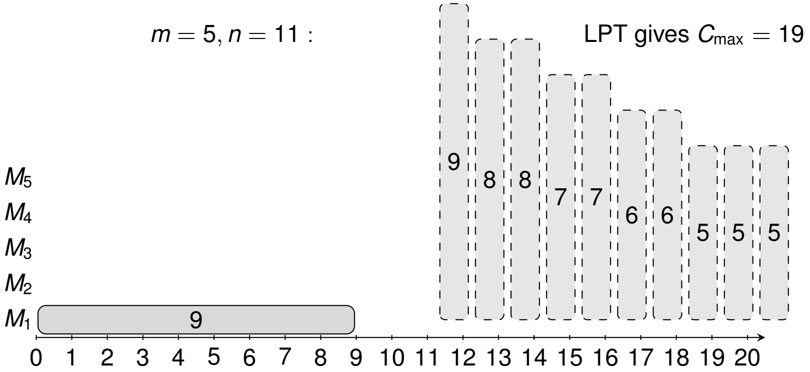

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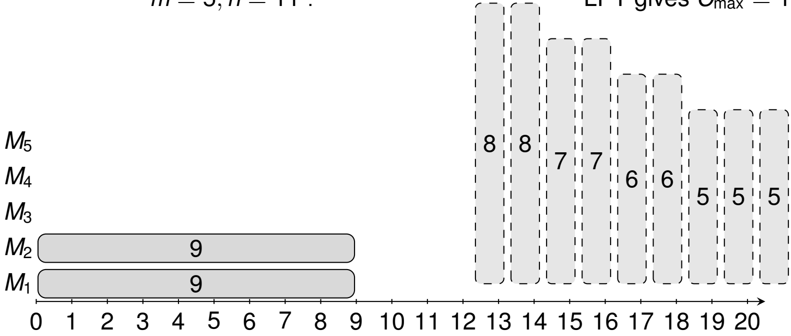

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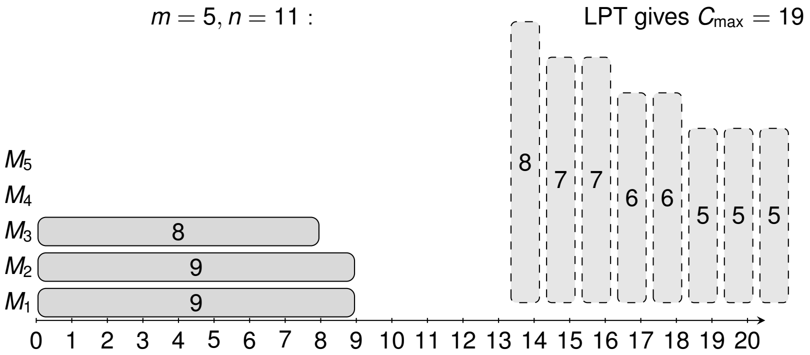

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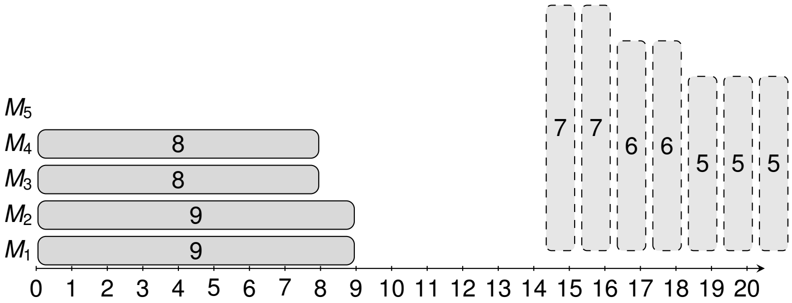

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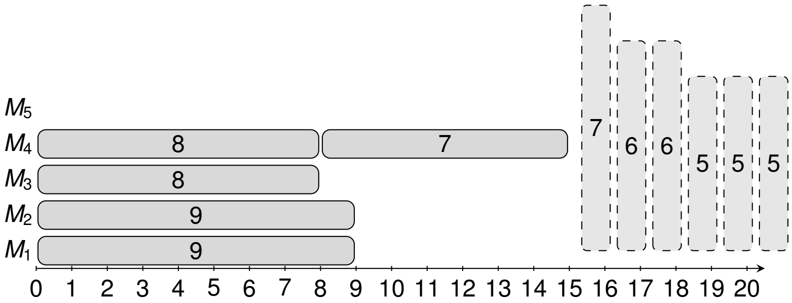

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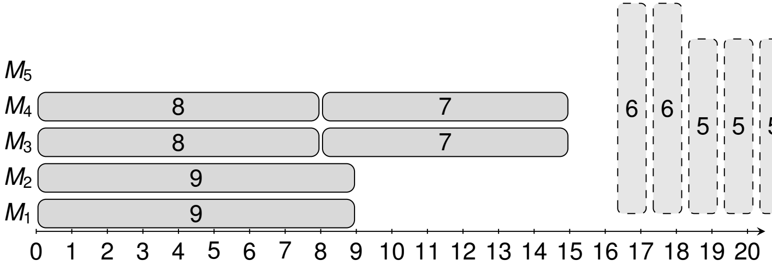

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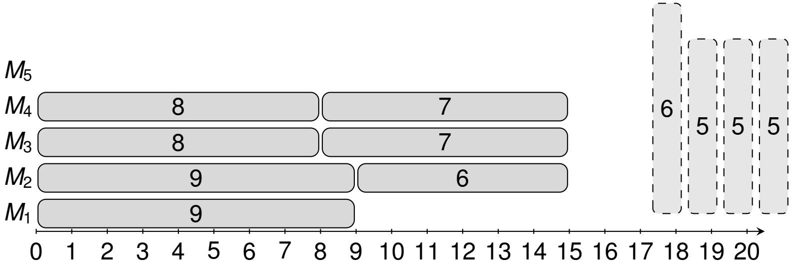

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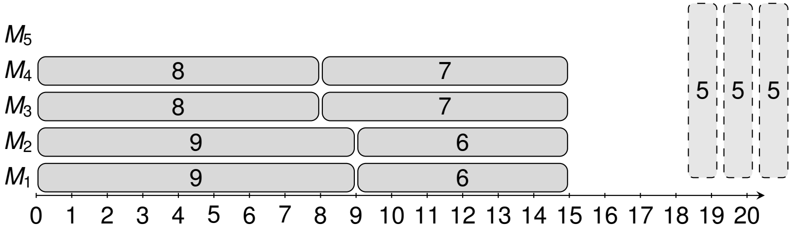

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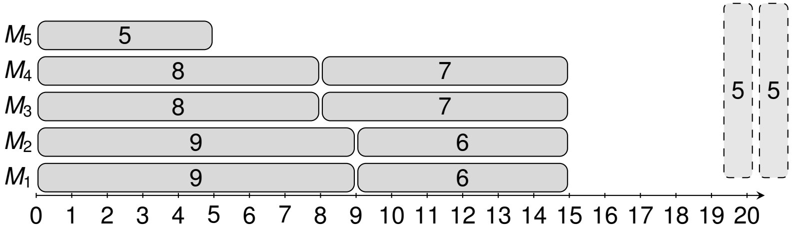

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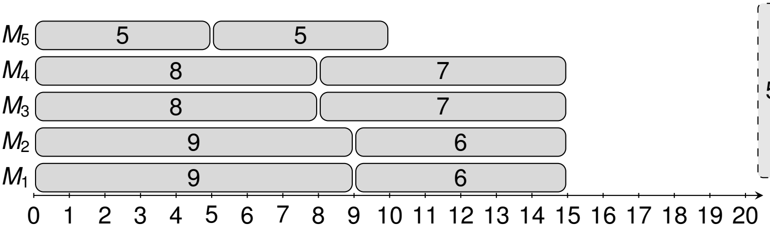

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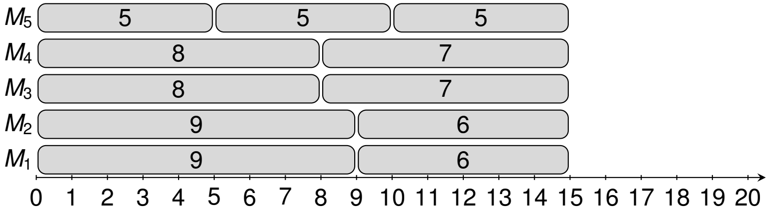

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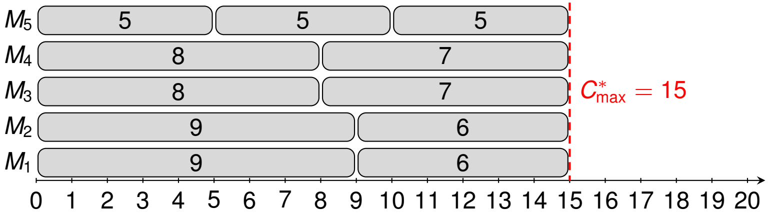

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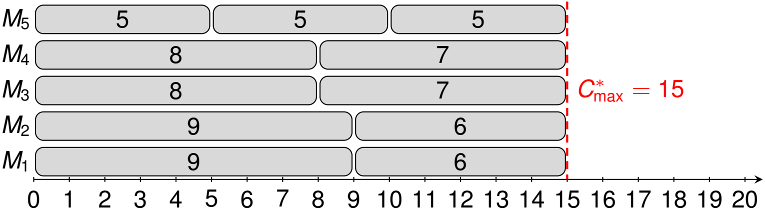
Optimum is $C^*_{\max} = 15$



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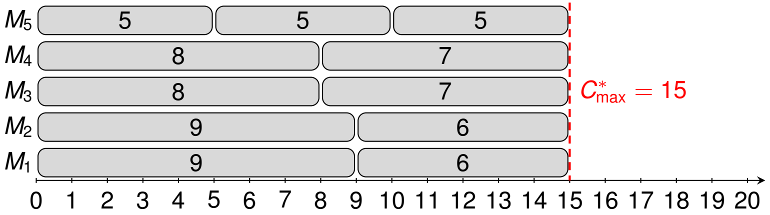
$m = 5, n = 11$:

LPT gives $C_{\max} = 19$

Optimum is $C^*_{\max} = 15$



A PTAS for Parallel Machine Scheduling

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SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

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There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

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PTAS$(J_1, J_2, \ldots, J_n, m)$
1: Do binary search to find smallest T s.t. $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.
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the “well-known” formula

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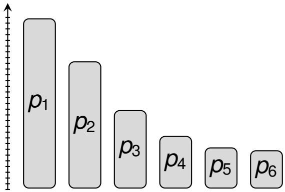

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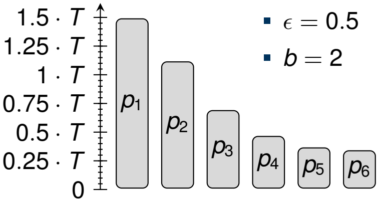

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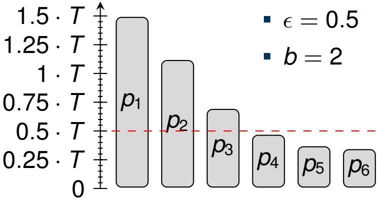

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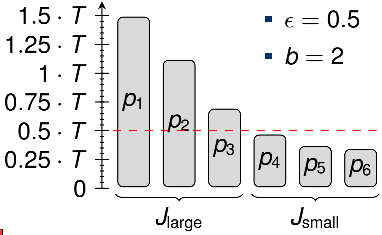

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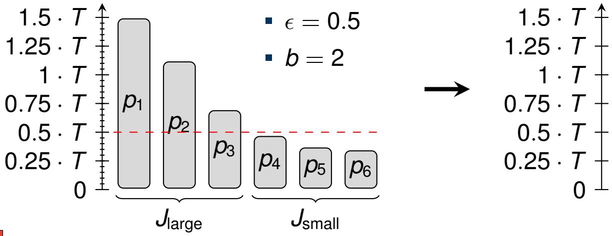

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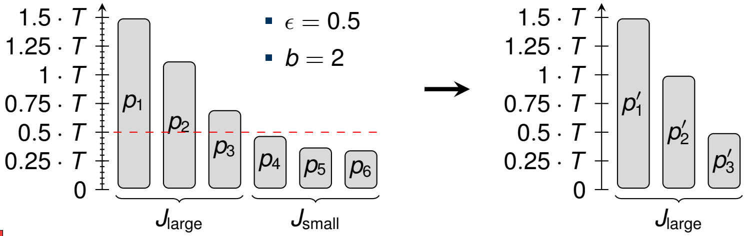

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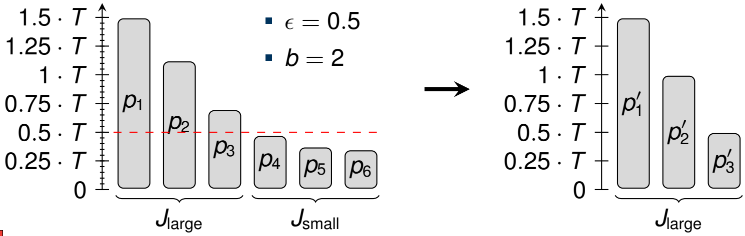
$\Rightarrow$ Every $p'_i = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

Can assume there are no jobs with $p_j \geq T$!



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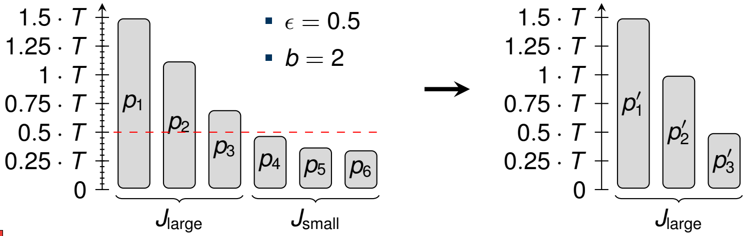

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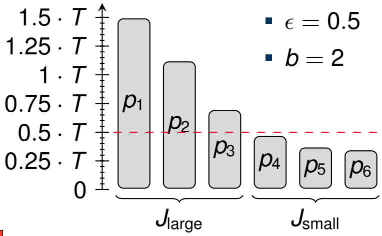
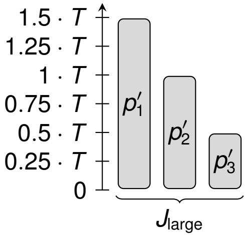
Assignments to one machine with makespan $\leq T$.





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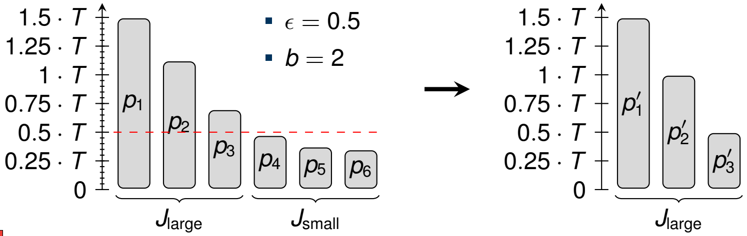

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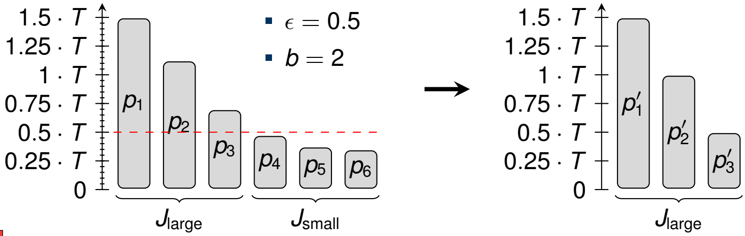

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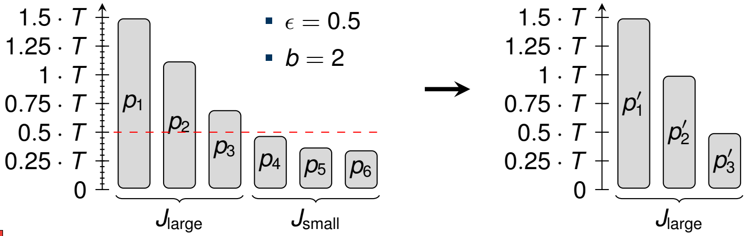

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Assign some jobs to one machine, and then use as few machines as possible for the rest.

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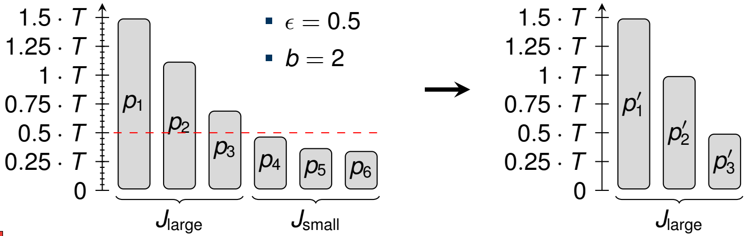

- $\epsilon = 0.5$
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List scheduling has an approximation ratio of 2.

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The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

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Because for sufficiently small approximation ratio $1 + \epsilon$, the computed solution has to be optimal, and Parallel Machine Scheduling is strongly NP-hard.

VI. Approximation Algorithms: Travelling Salesman Problem

Thomas Sauerwald

Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

General TSP

Metric TSP

The Traveling Salesman Problem (TSP)

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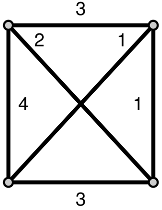

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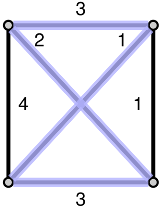

$3 + 2 + 1 + 3 = 9$

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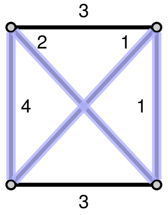

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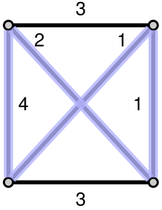

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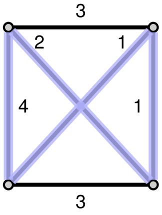

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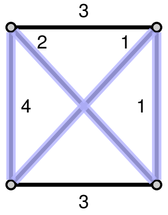

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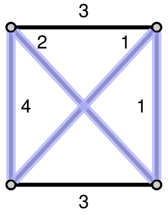

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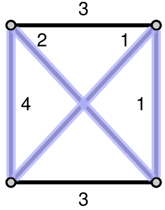

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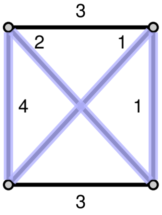

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Even this version is NP hard (Ex. 35.2-2)

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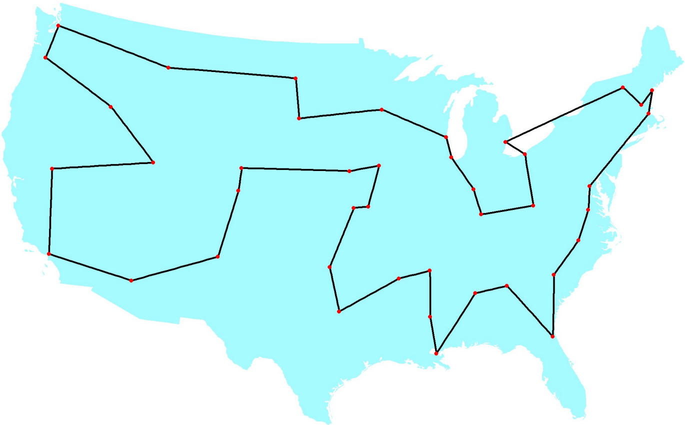
History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



`http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html`

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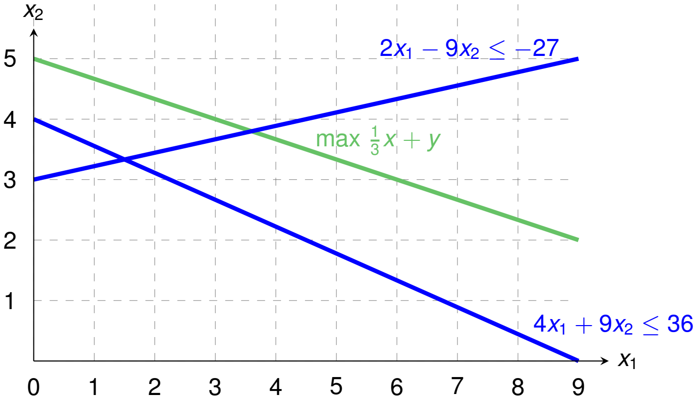

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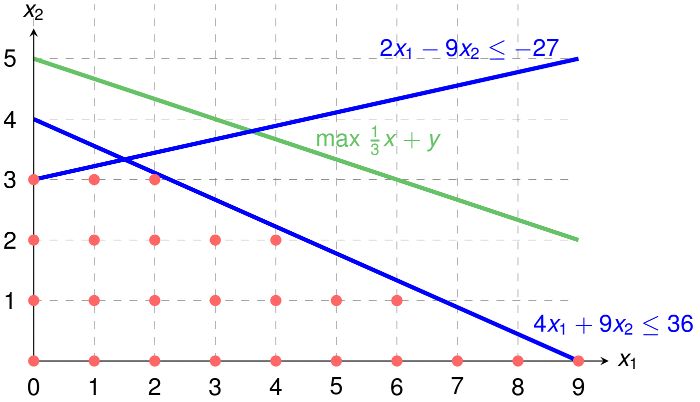

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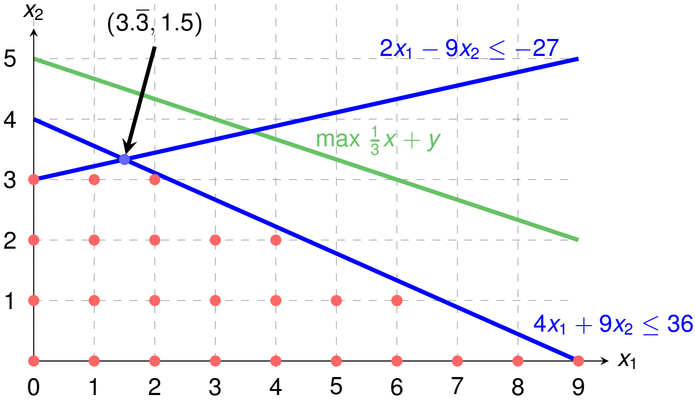

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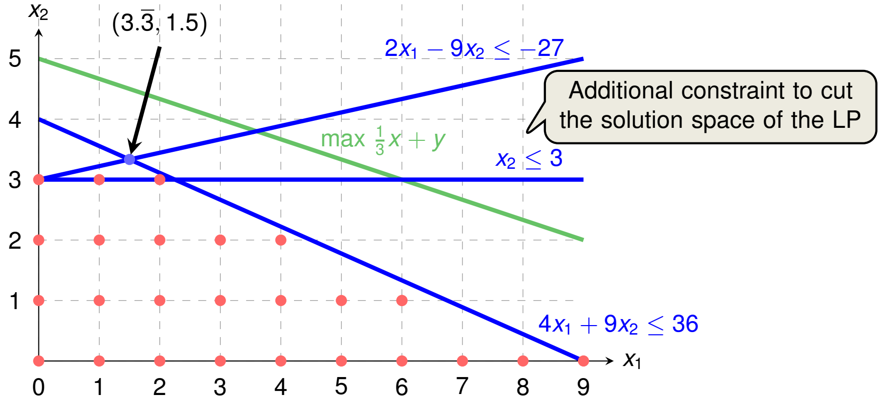

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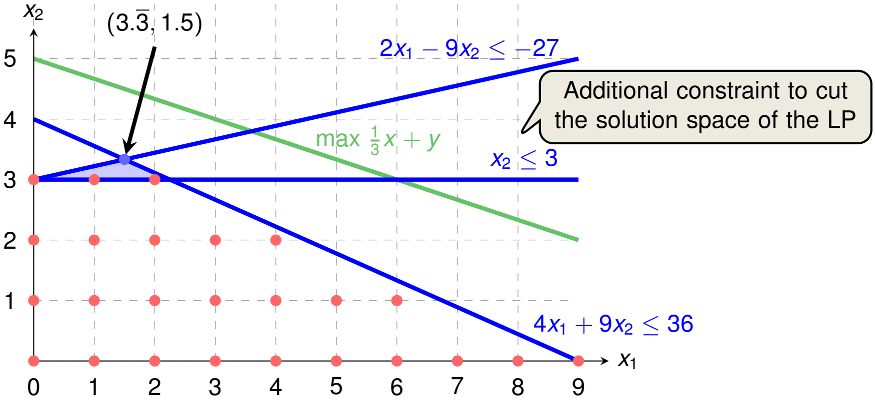

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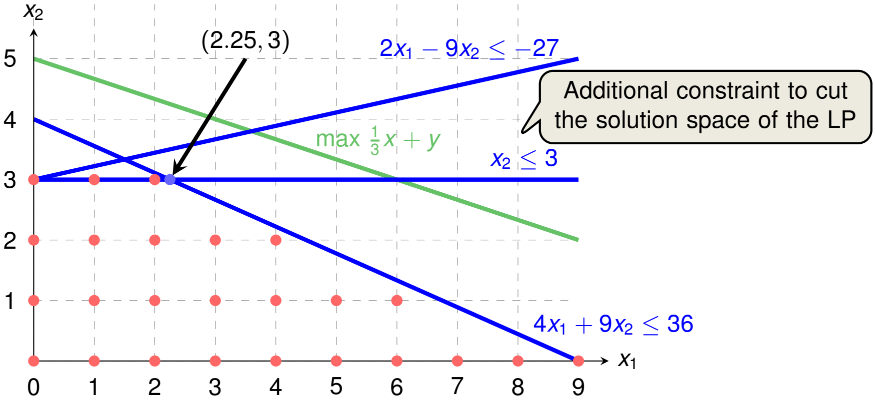

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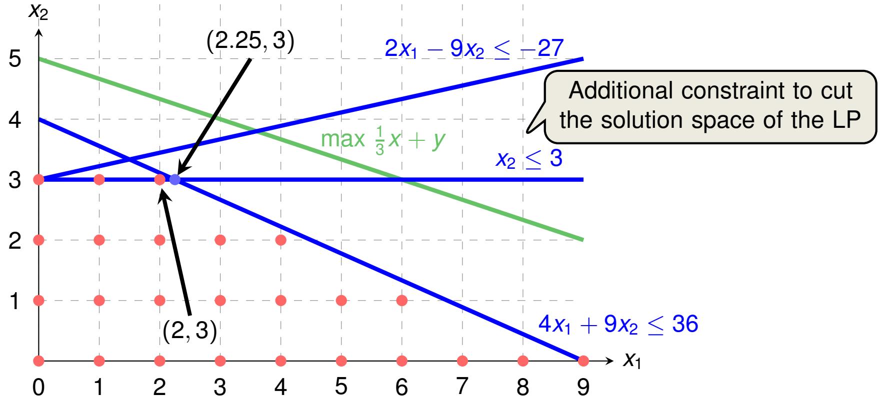

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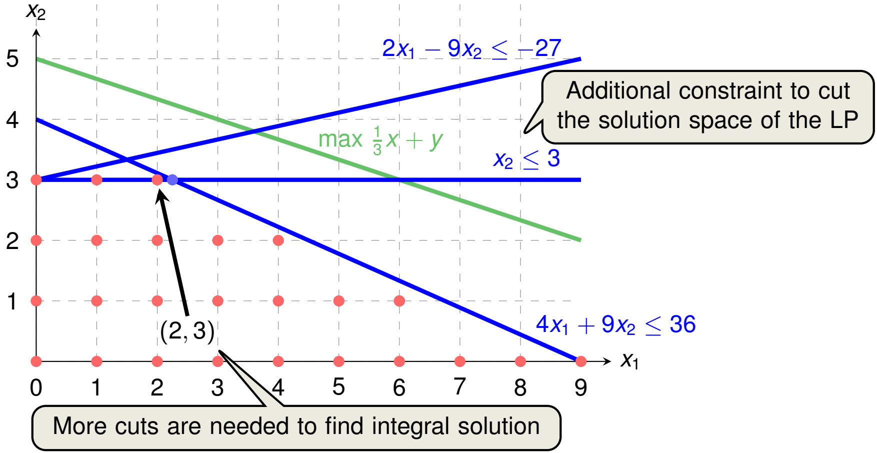

Outline

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General TSP

Metric TSP

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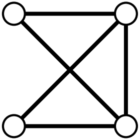

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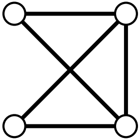

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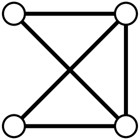

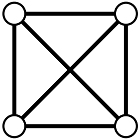

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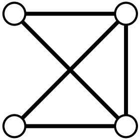
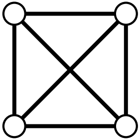
$G = (V, E)$ 

 $G' = (V, E')$

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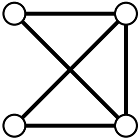

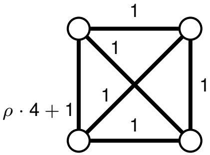

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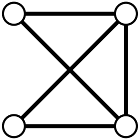
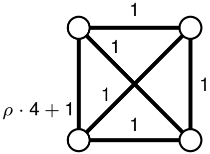
Large weight will render this edge useless!

$G = (V, E)$





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Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

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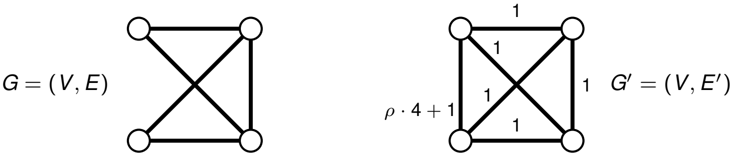

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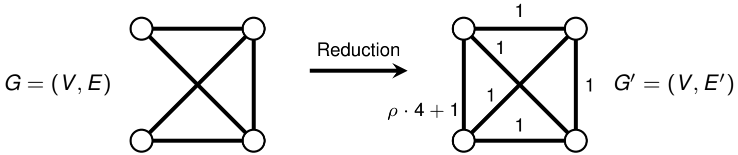

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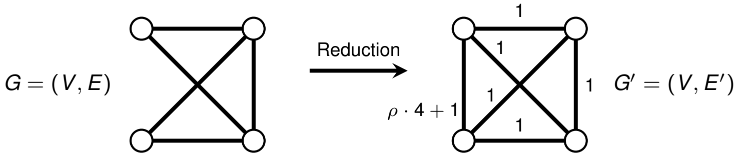

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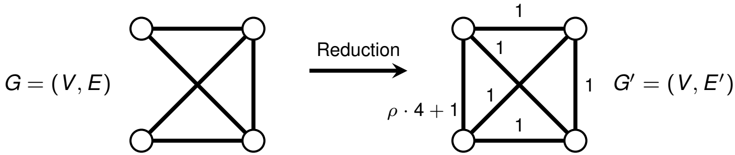

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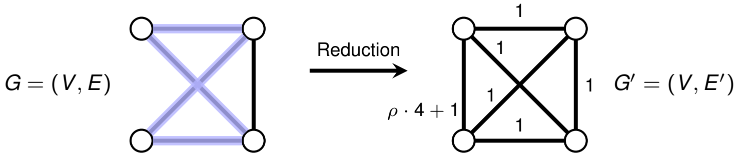

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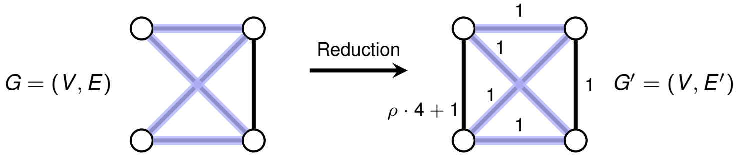

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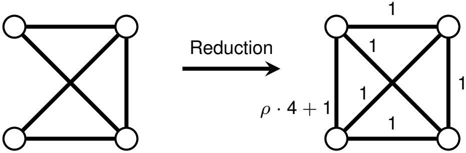

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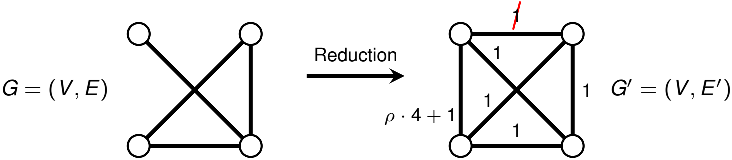

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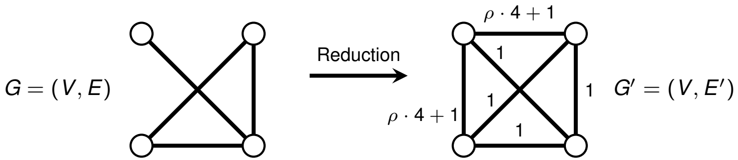

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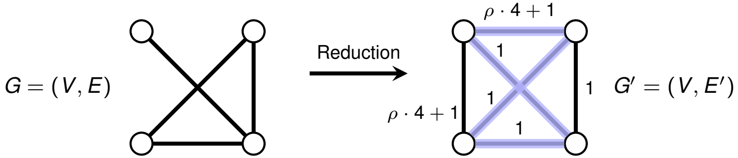

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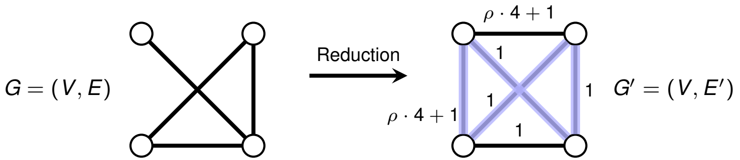

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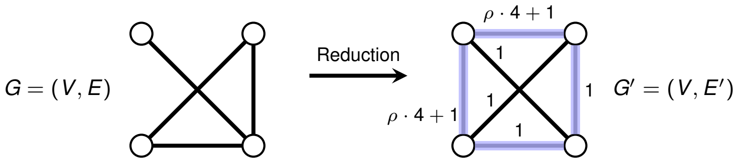

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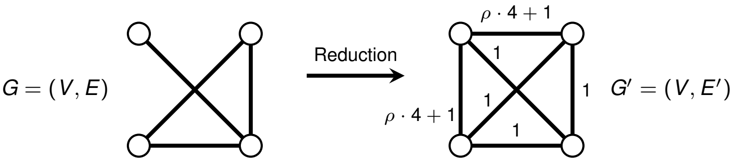

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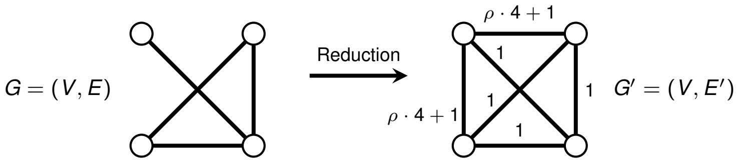

Hardness of Approximation

Theorem 35.3

If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general TSP.

Idea: Reduction from the hamiltonian-cycle problem.

Proof:

- Let $G = (V, E)$ be an instance of the hamiltonian-cycle problem
- Let $G' = (V, E')$ be a complete graph with costs for each $(u, v) \in E'$:

$$c(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E, \\ \rho|V| + 1 & \text{otherwise.} \end{cases}$$

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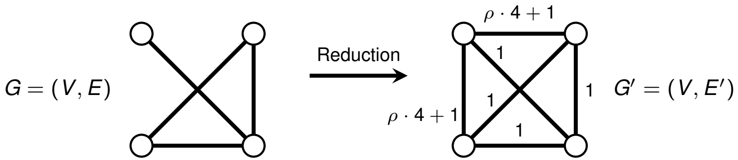

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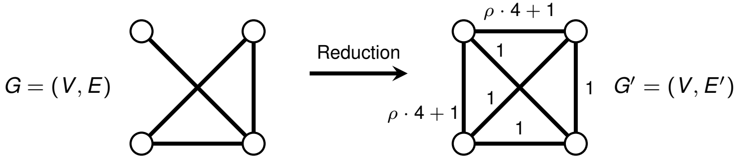

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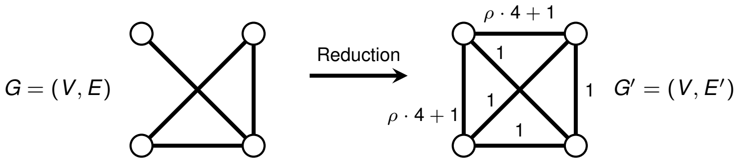
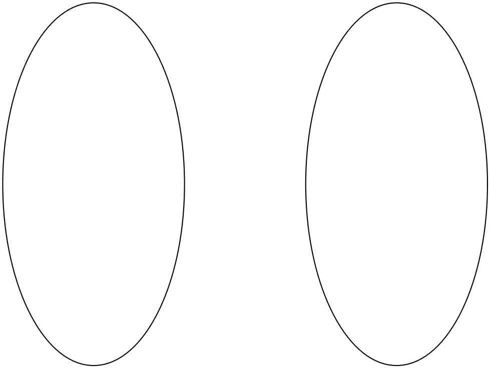


Proof of Theorem 35.3 from a higher perspective



instances of Hamilton instances of TSP

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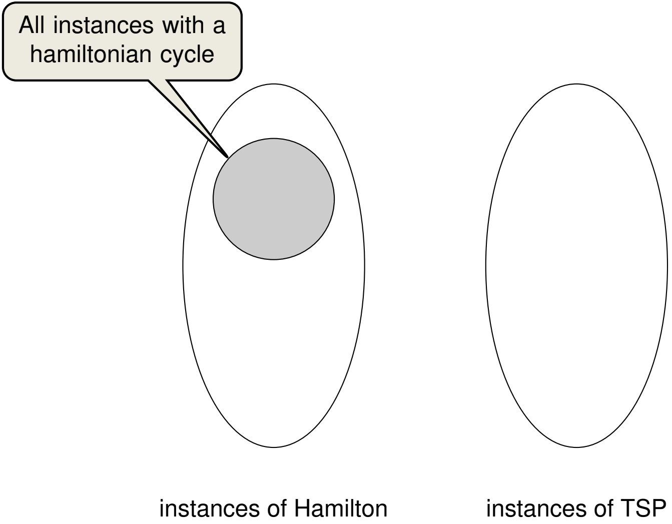

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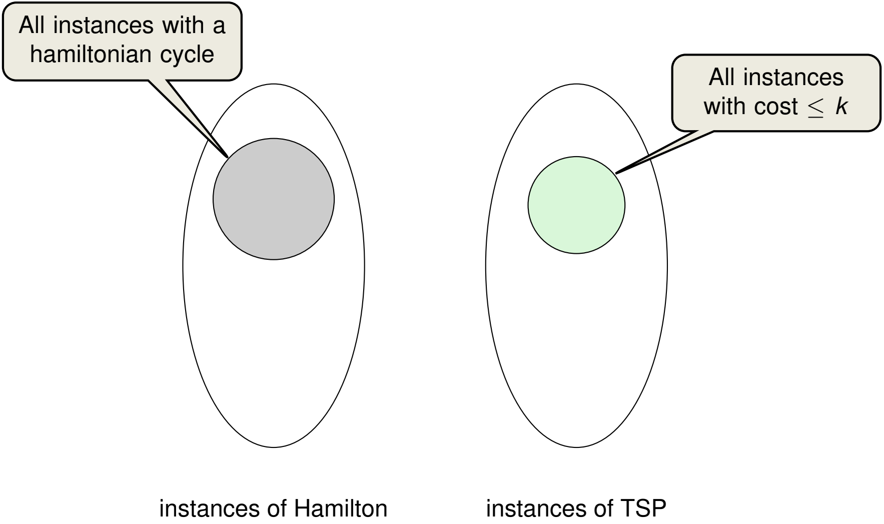

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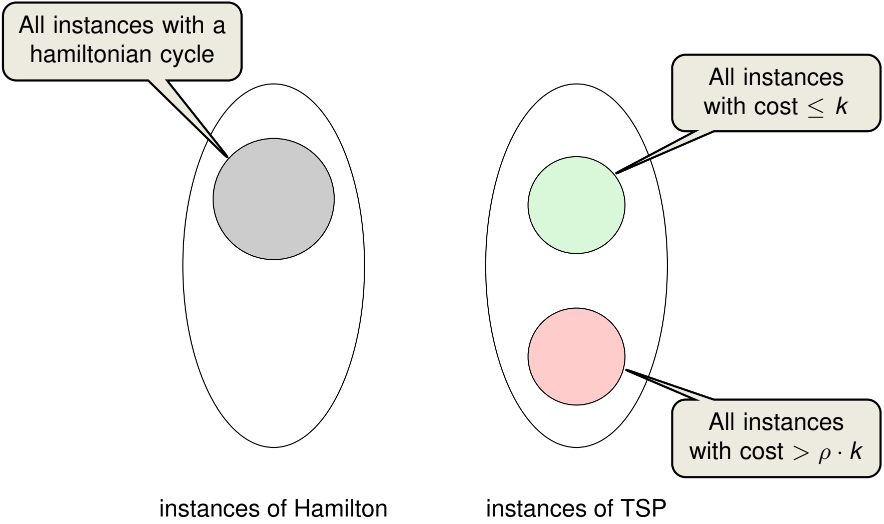

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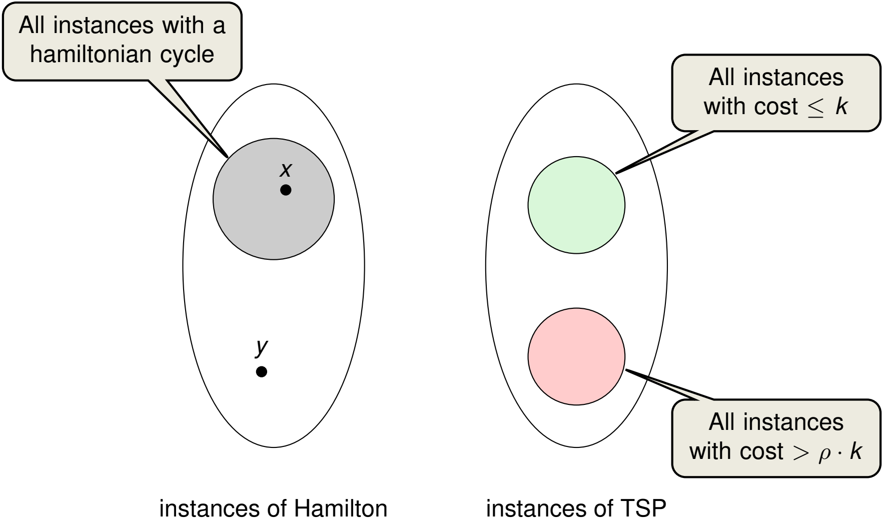

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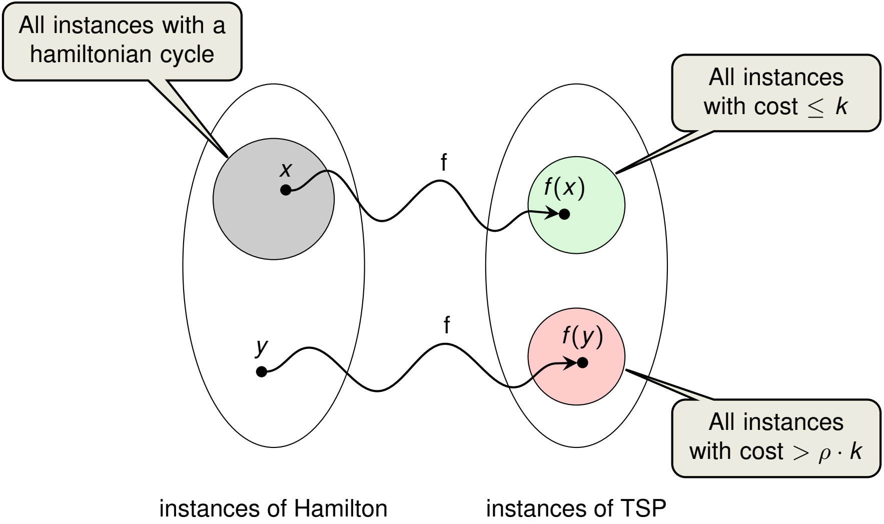

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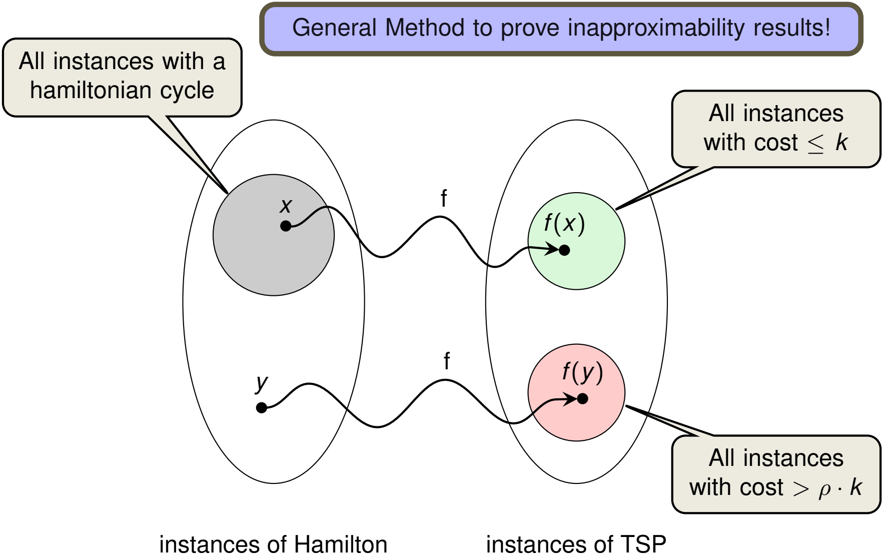

Outline

Introduction

General TSP

Metric TSP

Metric TSP (TSP Problem with the Triangle Inequality)

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR(G, c)
select a vertex $r \in G.V$ to be a “root” vertex
compute a minimum spanning tree T_{min} for G from root r
    using MST-PRIM(G, c, r)
let H be a list of vertices, ordered according to when they are first visited
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return the hamiltonian cycle H

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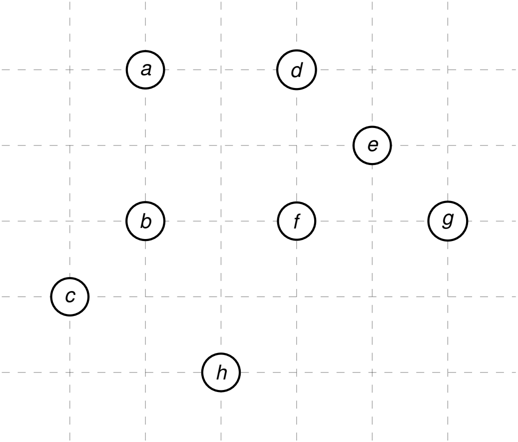
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



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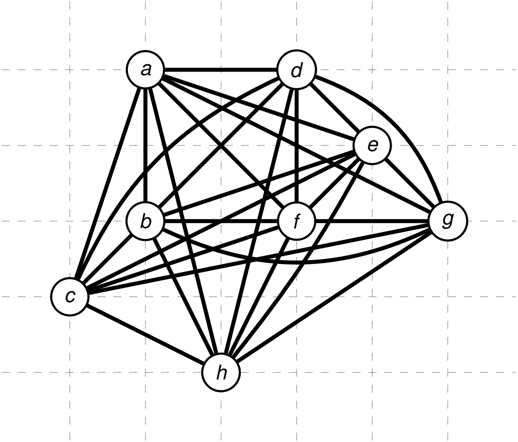
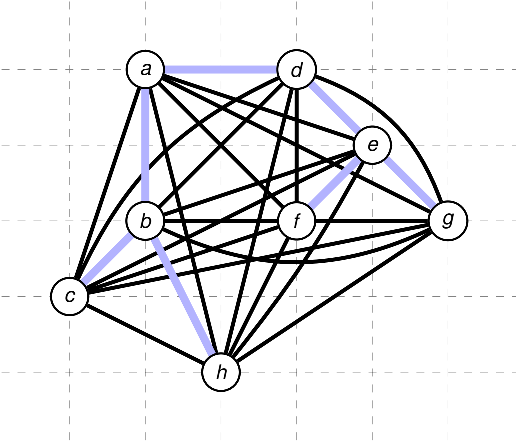


1. Compute MST $T_{\min}$

Run of Approx-Tsp-Tour



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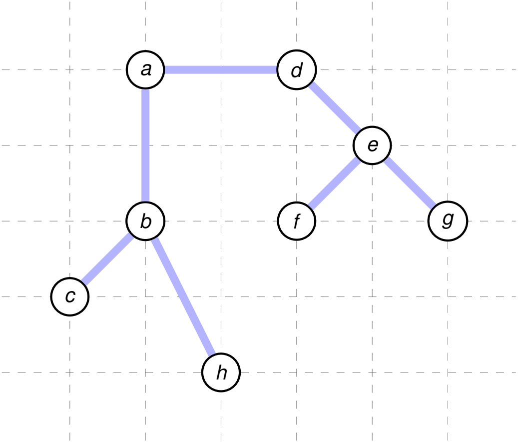

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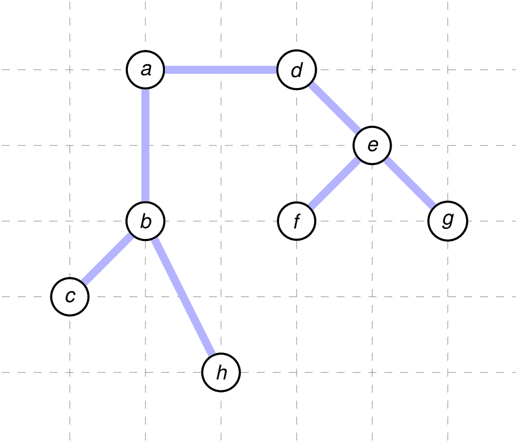

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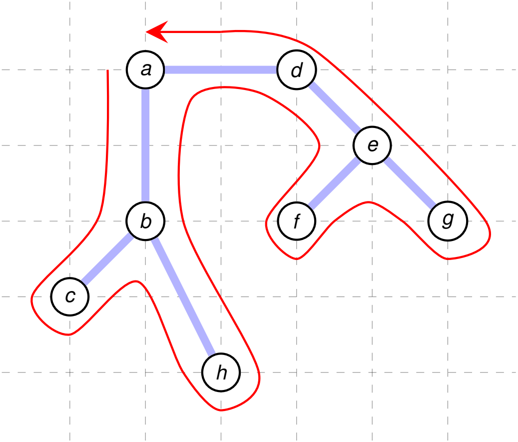

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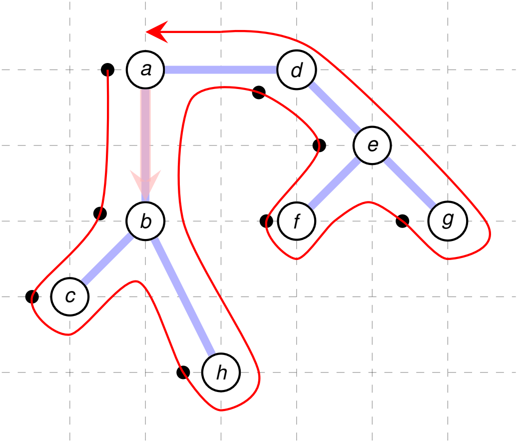

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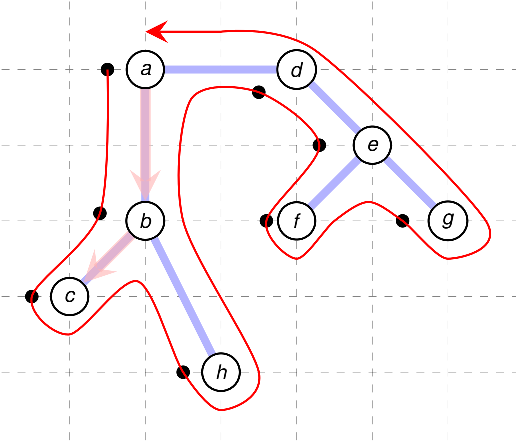

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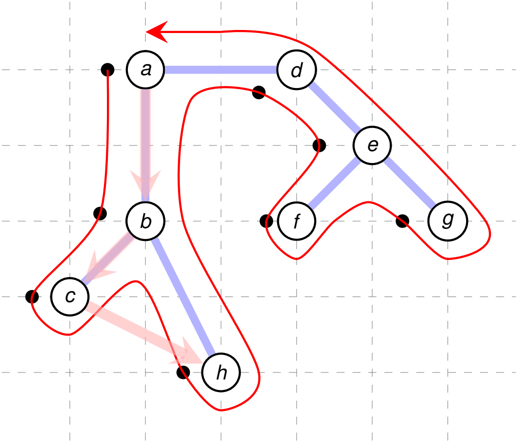

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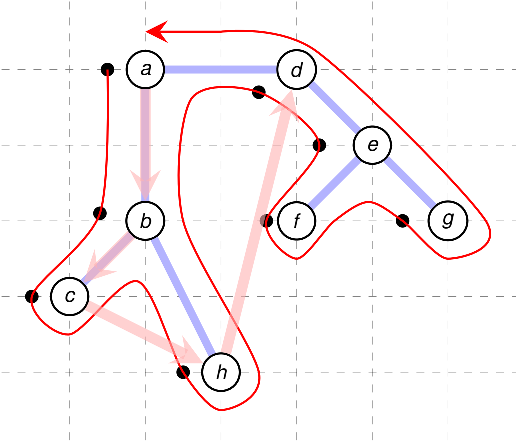

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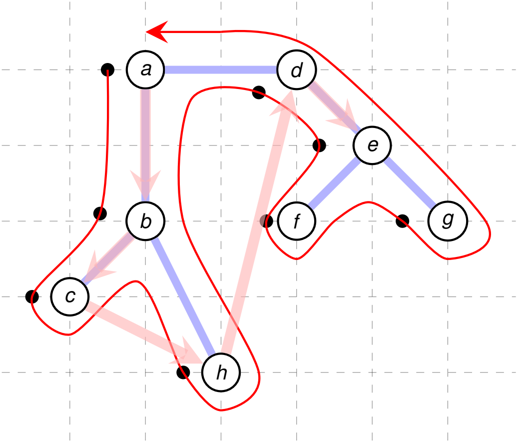

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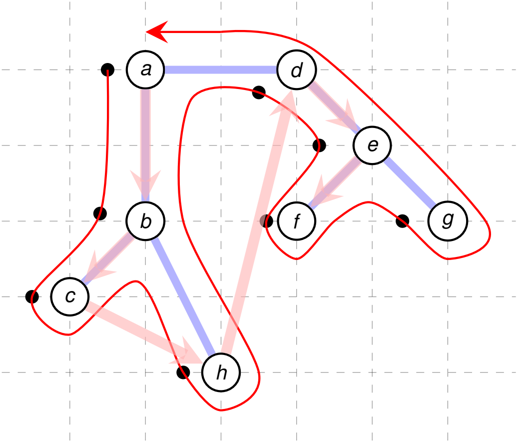

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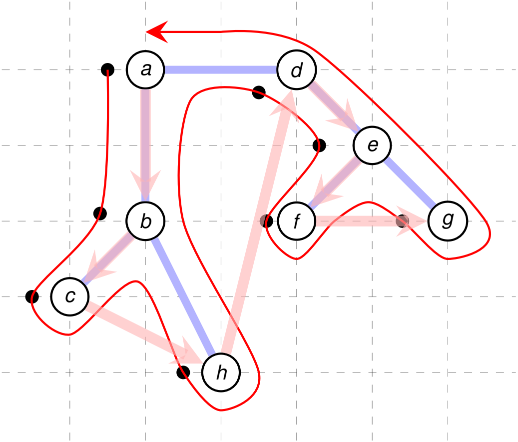

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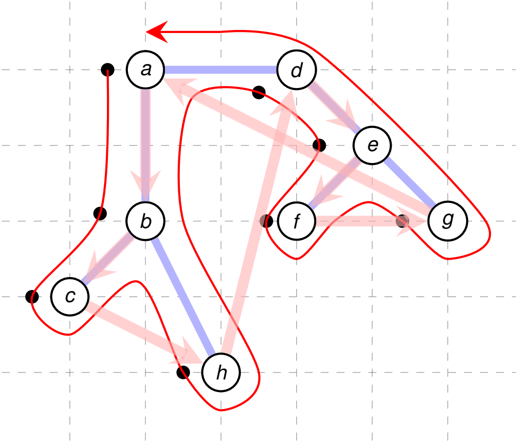

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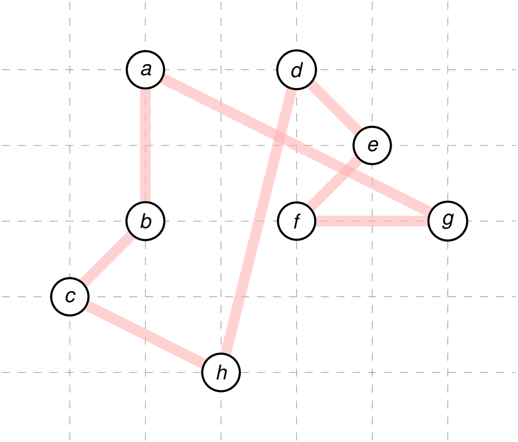

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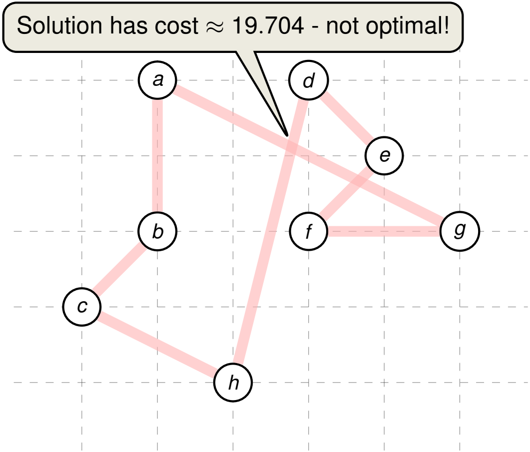

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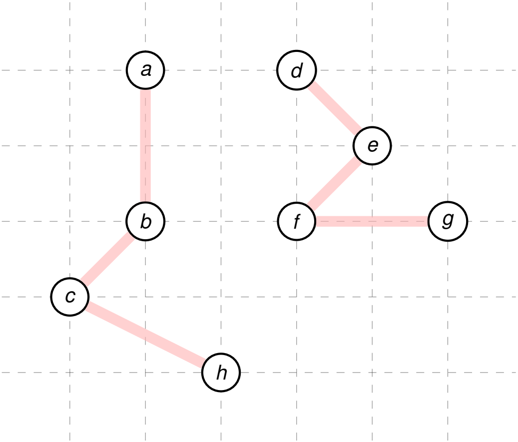

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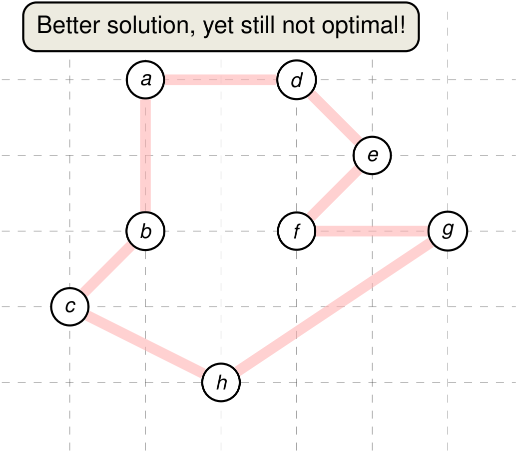

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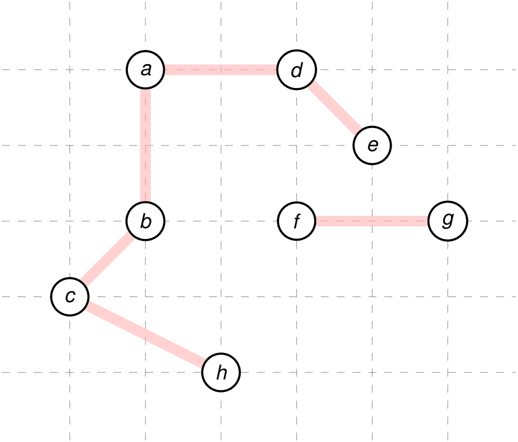

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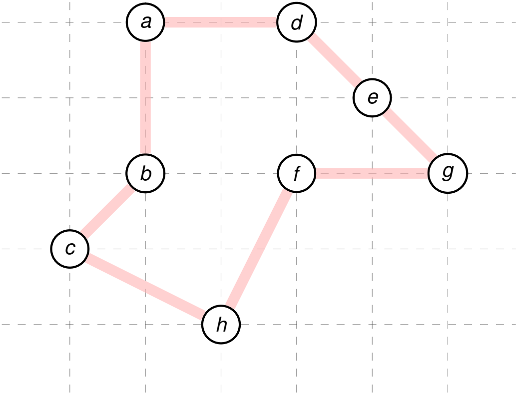
This is the optimal solution (cost $\approx$ 14.715).



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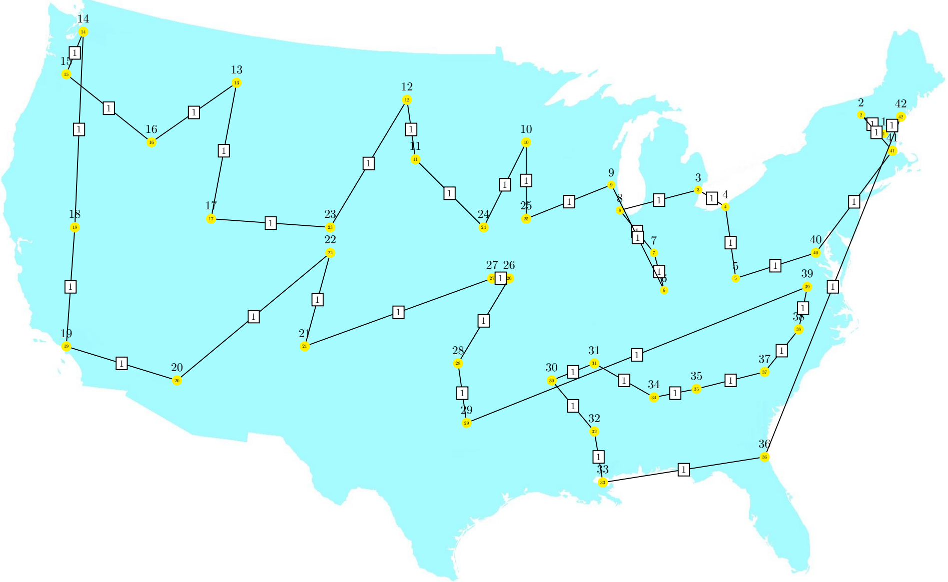
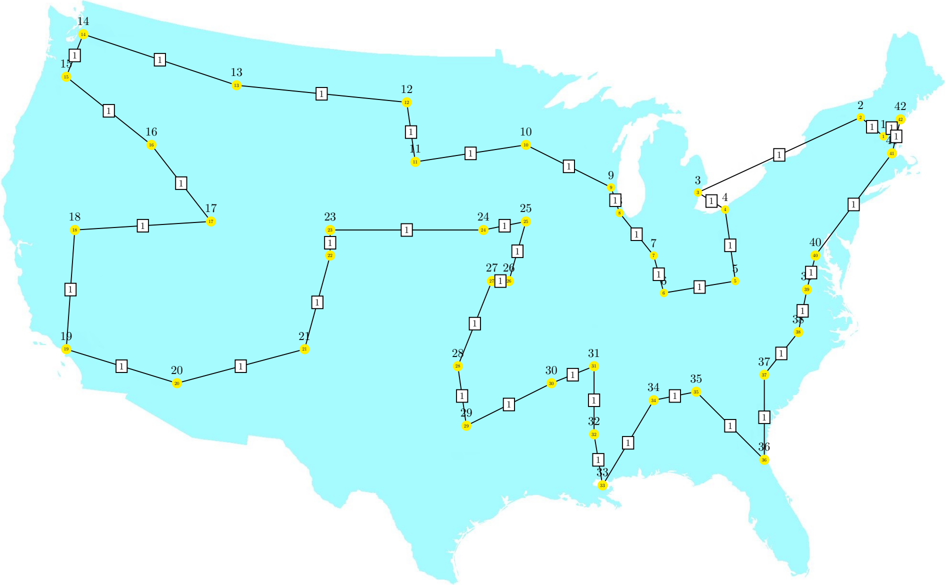
Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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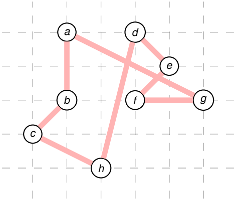

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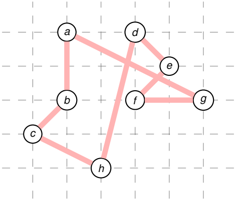

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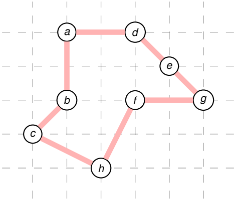

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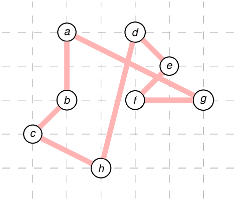

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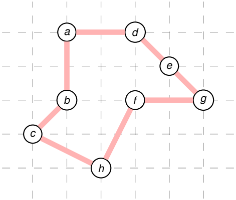

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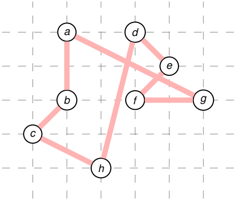

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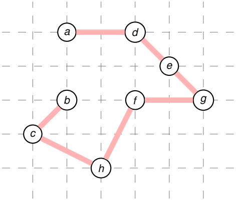

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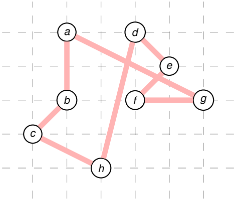

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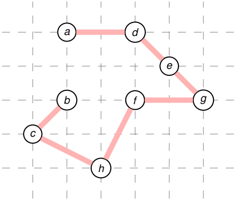

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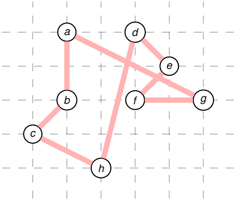

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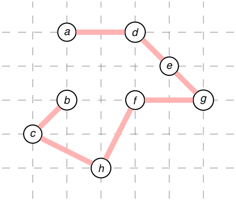

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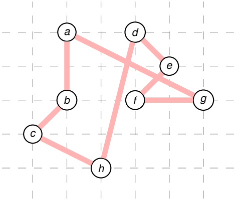
exploiting that all edge costs are non-negative!



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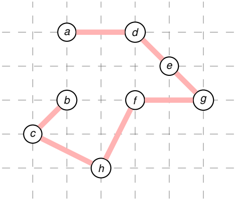

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Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

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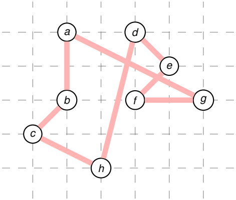
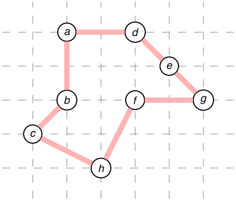


solution H of APPROX-TSP



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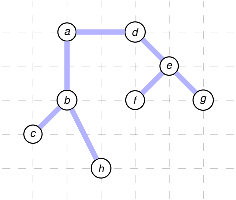
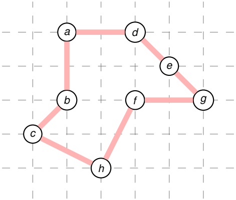


minimum spanning tree $T_{\min}$



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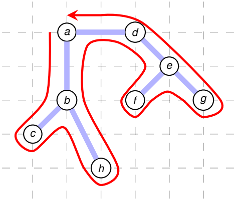

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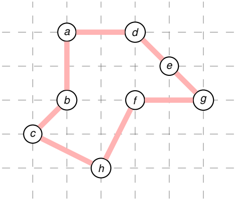

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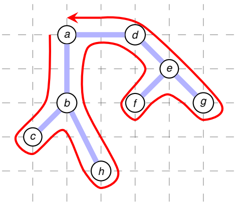

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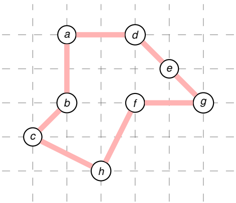

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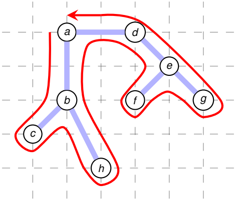

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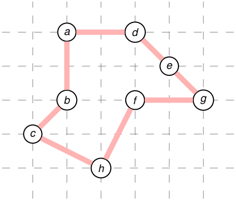

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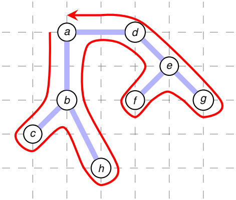

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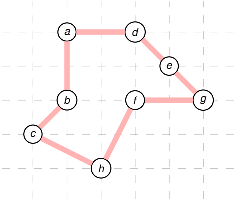

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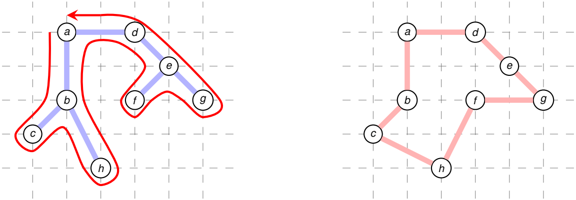

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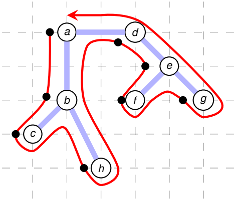

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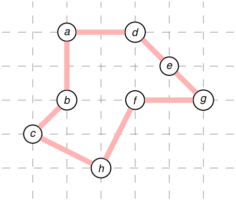

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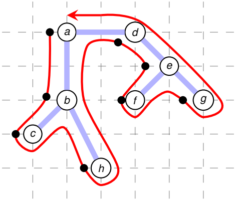

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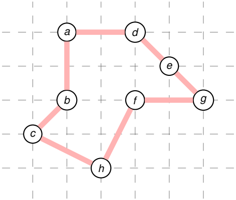

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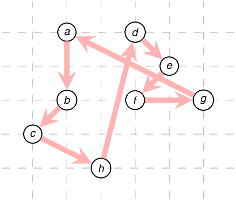
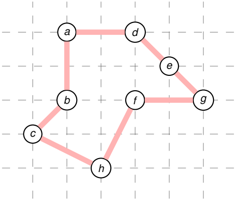


Tour $H = (a, b, c, h, d, e, f, g, a)$



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exploiting triangle inequality!

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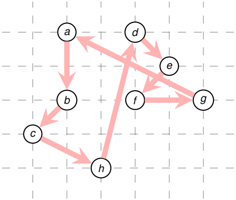

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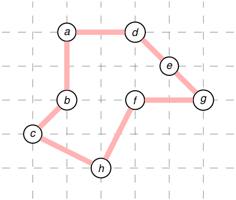

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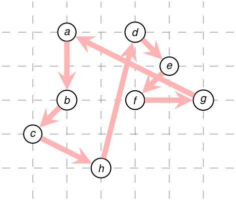

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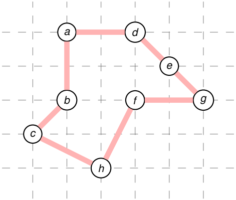

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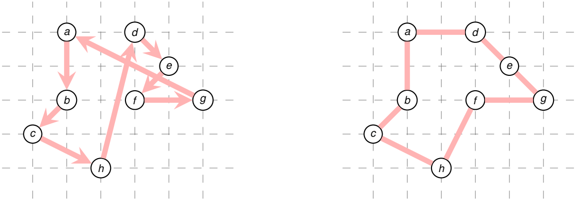

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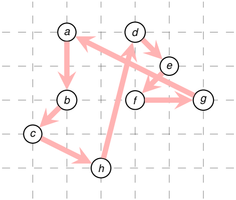

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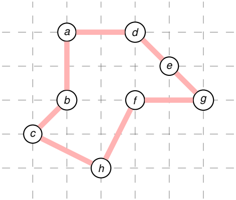

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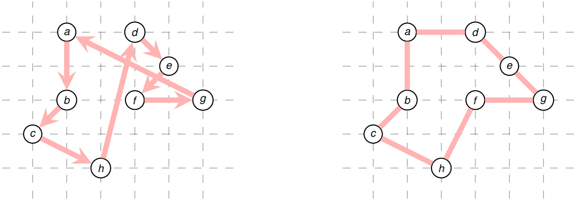
$$c(H) \leq c(W) \leq 2c(H^*) \qquad \square$$



Tour $H = (a, b, c, h, d, e, f, g, a)$ — optimal solution H^*

Christofides Algorithm

Theorem 35.2

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Christofides Algorithm

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APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Can we get a better approximation ratio?

Christofides Algorithm

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Can we get a better approximation ratio?

CHRISTOFIDES(G, c)
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree $T_{\min}$ for G from root r
    using MST-PRIM(G, c, r)
compute a perfect matching $M_{\min}$ with minimum weight in the complete graph
    over the odd-degree vertices in $T_{\min}$
let H be a list of vertices, ordered according to when they are first visited
    in a Eulearian circuit of $T_{\min} \cup M_{\min}$
return the hamiltonian cycle H

Christofides Algorithm

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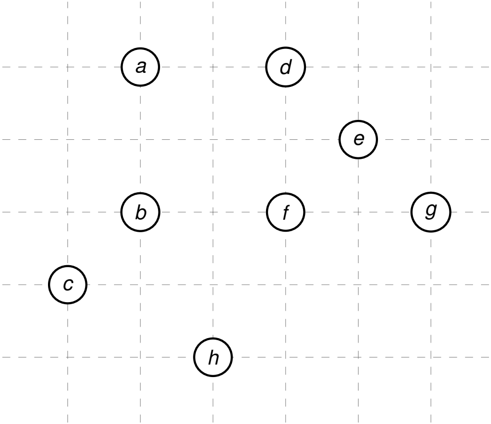
Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of CHRISTOFIDES



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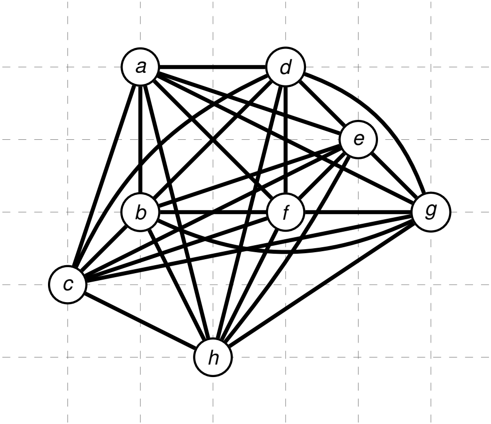
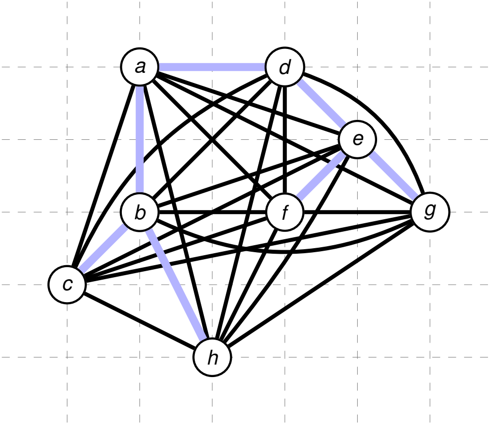


1. Compute MST T_{min}

Run of CHRISTOFIDES



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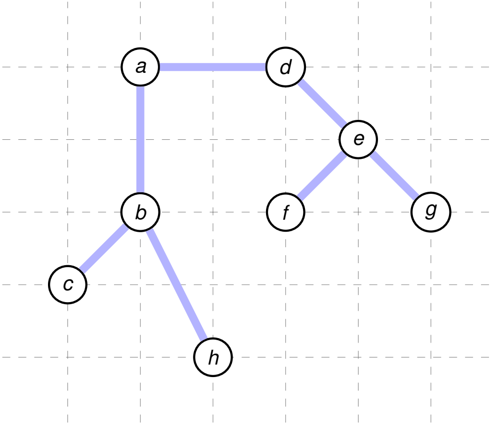
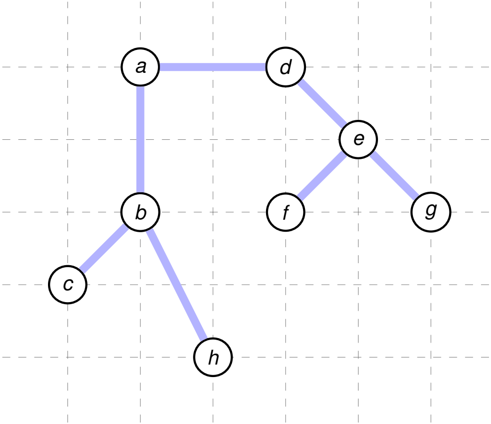
Run of CHRISTOFIDES



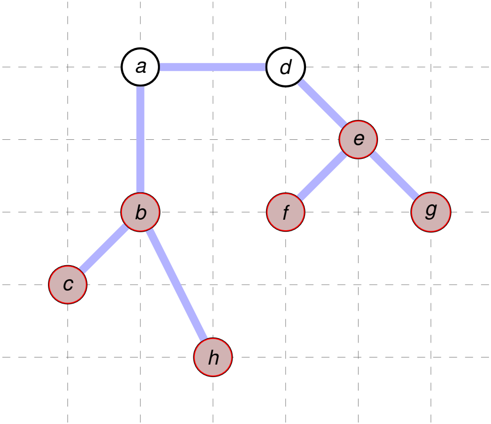
1. Compute MST $T_{\min}$ ✓

Run of CHRISTOFIDES



1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$

Run of CHRISTOFIDES



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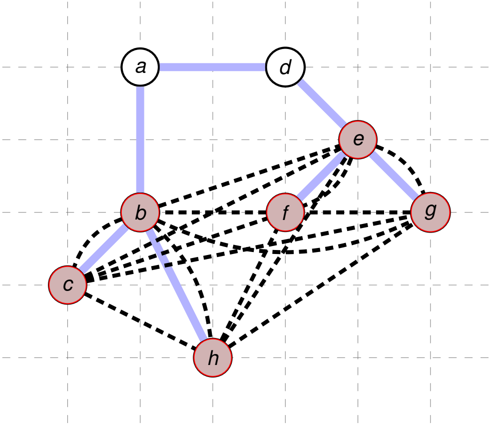

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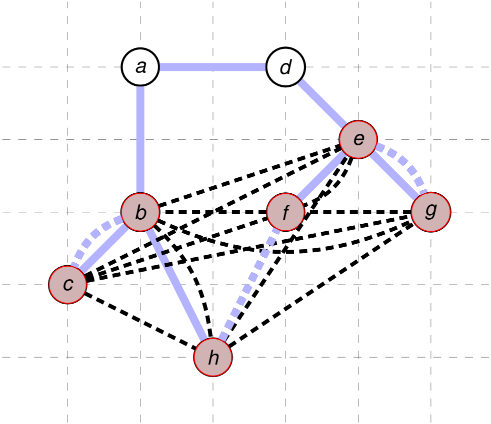
Run of CHRISTOFIDES



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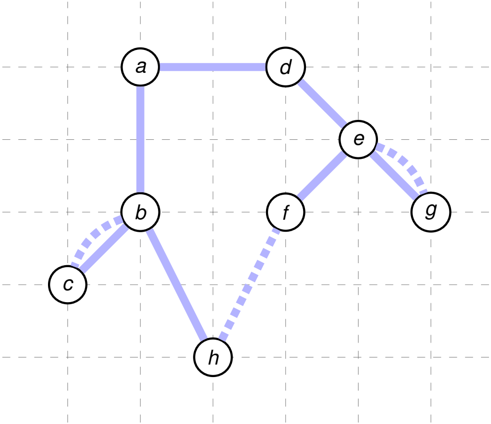

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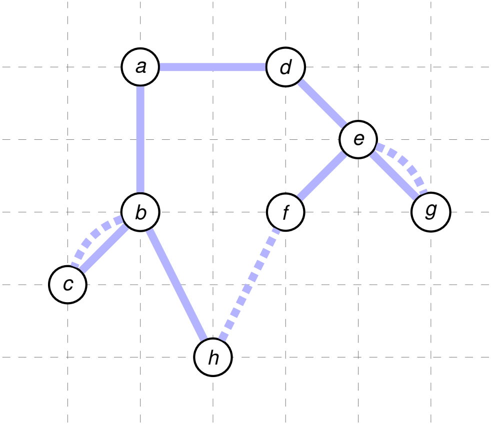
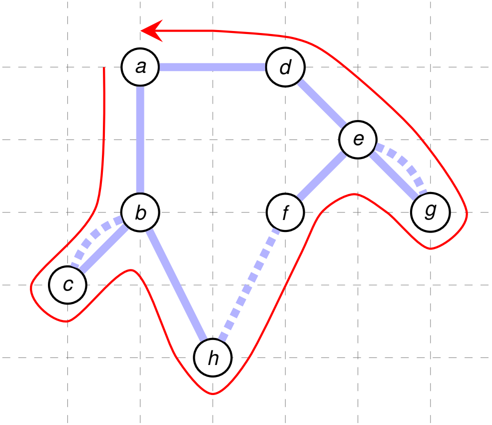
Run of CHRISTOFIDES



1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$ ✓
3. Find an Eulerian Circuit in $T_{\min} \cup M_{\min}$

All vertices in $T_{\min} \cup M_{\min}$ have even degree!

Run of CHRISTOFIDES



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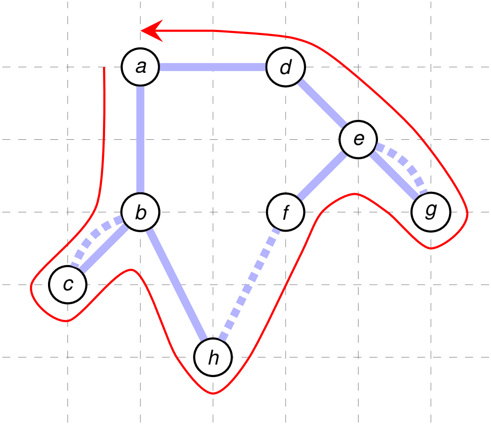
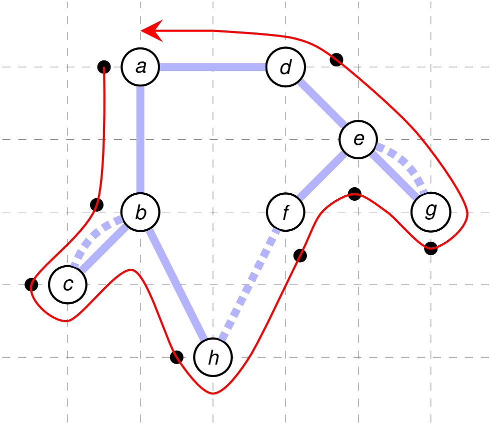


1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$ ✓
3. Find an Eulerian Circuit in $T_{\min} \cup M_{\min}$ ✓
4. Transform the Circuit into a Hamiltonian Cycle

Run of CHRISTOFIDES



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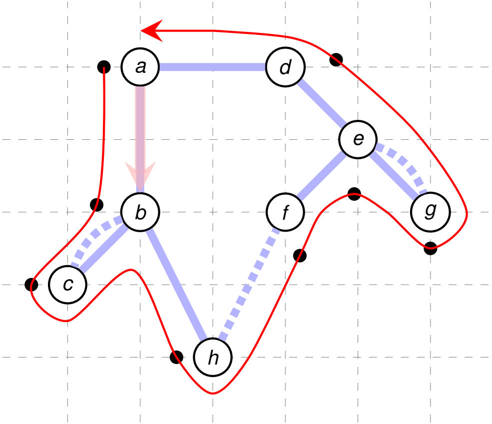

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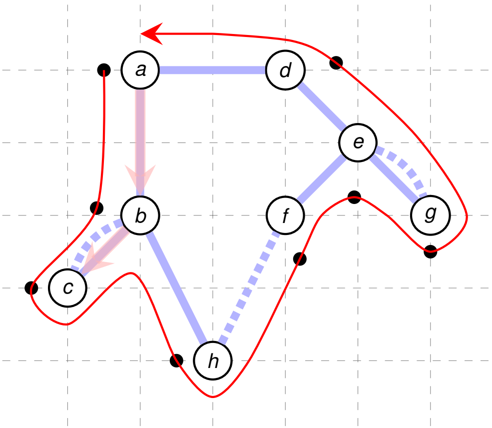

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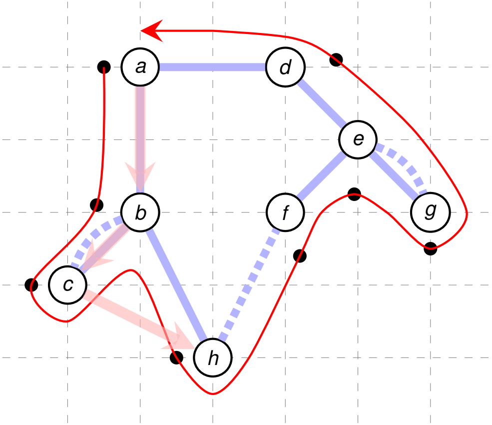

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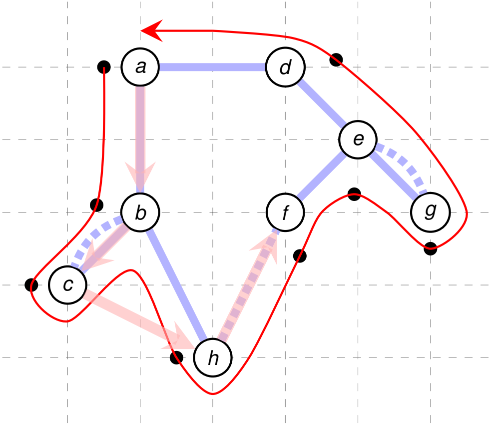

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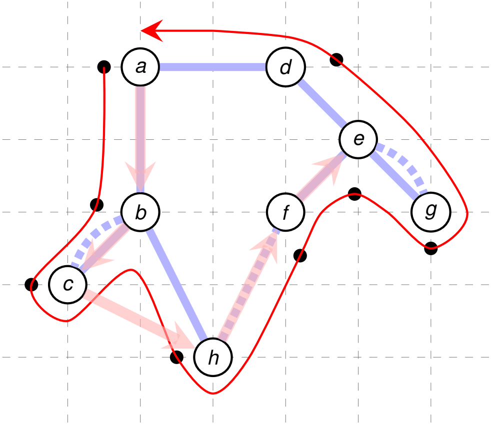

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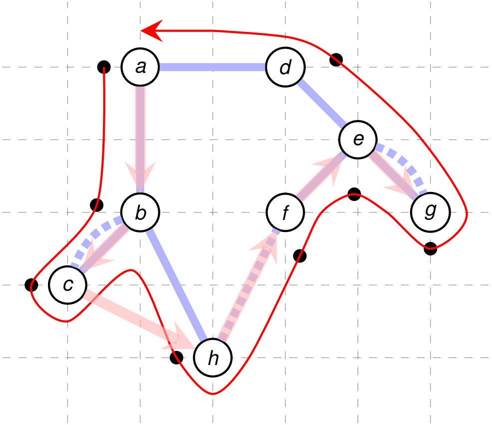

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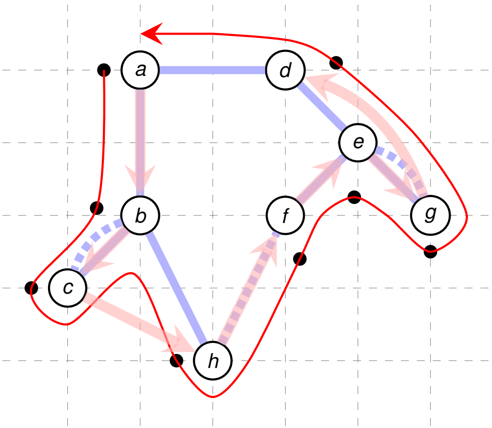

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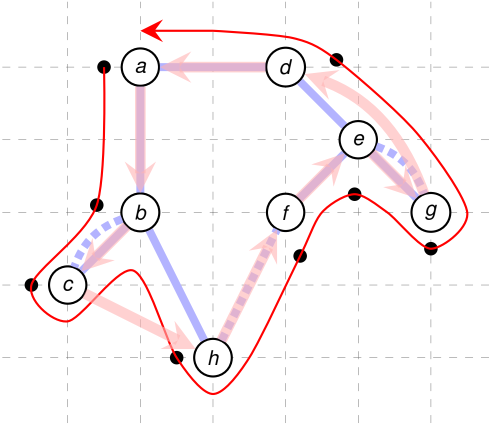

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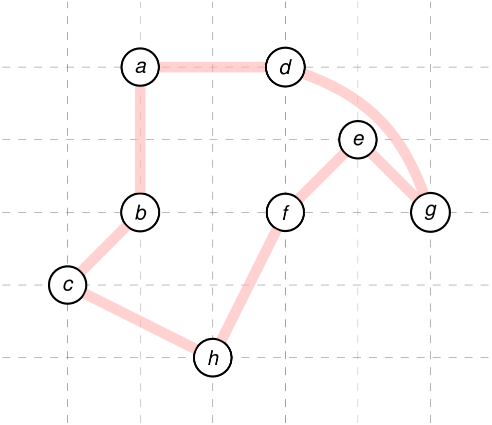

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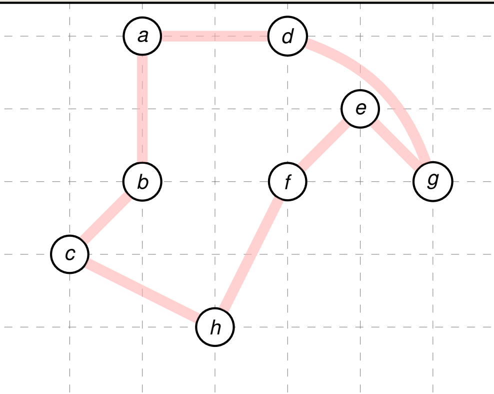
Run of CHRISTOFIDES

Solution has cost $\approx$ 15.54 - within 10% of the optimum!



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Proof of the Approximation Ratio

Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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- Combining 1 with 2 yields

$$c(W) \leq c(H^*) + c(M_{\min}) \leq c(H^*) + \frac{1}{2} c(H^*) = \frac{3}{2} c(H^*).$$

Concluding Remarks

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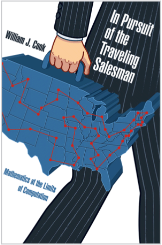


VII. Approximation Algorithms: Randomisation and Rounding

Thomas Sauerwald

Easter 2018

UNIVERSITY OF CAMBRIDGE

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

Performance Ratios for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost C of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

Call such an algorithm randomised $\rho(n)$-approximation algorithm.

extends in the natural way to randomised algorithms

Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-3-CNF Satisfiability

Assume that no literal (including its negation) appears more than once in the same clause.

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

Relaxation of the satisfiability problem. Want to compute how "close" the formula to being satisfiable is.

Example:

$$(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_2 \vee \overline{x_4} \vee x_5) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$

$x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ and $x_5 = 1$ satisfies 3 (out of 4 clauses)

Idea: What about assigning each variable independently at random?

Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

Proof:

- For every clause $i = 1, 2, \ldots, m$, define a random variable:
$$Y_i = \mathbf{1}\{\text{clause } i \text{ is satisfied}\}$$
- Since each literal (including its negation) appears at most once in clause i,
$$\mathbf{Pr}[\,\text{clause } i \text{ is not satisfied}\,] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$
$$\Rightarrow \qquad \mathbf{Pr}[\,\text{clause } i \text{ is satisfied}\,] = 1 - \frac{1}{8} = \frac{7}{8}$$
$$\Rightarrow \qquad \mathbf{E}[\,Y_i\,] = \mathbf{Pr}[\,Y_i = 1\,] \cdot 1 = \frac{7}{8}.$$
- Let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,
$$\mathbf{E}[\,Y\,] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[\,Y_i\,] = \sum_{i=1}^{m} \frac{7}{8} = \frac{7}{8} \cdot m. \qquad \square$$

Linearity of Expectations

maximum number of satisfiable clauses is m

Interesting Implications

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

Corollary

For any instance of MAX-3-CNF, there exists an assigment which satisfies at least $\frac{7}{8}$ of all clauses.

There is $\omega \in \Omega$ such that $Y(\omega) \geq \mathbf{E}[\,Y\,]$

Probabilistic Method: powerful tool to show existence of a non-obvious property.

Corollary

Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.

Follows from the previous Corollary.

Expected Approximation Ratio

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised 8/7-approximation algorithm.

One could prove that the probability to satisfy $(7/8) \cdot m$ clauses is at least $1/(8m)$

$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$!

GREEDY-3-CNF(ϕ, n, m)

for $j = 1, 2, \ldots, n$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 1\,]$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 0\,]$
    Let $x_j = v_j$ so that the conditional expectation is maximized
return the assignment $v_1, v_2, \ldots, v_n$

Analysis of GREEDY-3-CNF(ϕ, n, m)

This algorithm is deterministic.

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Proof:

- **Step 1:** polynomial-time algorithm
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over 2^{n-j+1} assignments
  - A smarter way is to use linearity of (conditional) expectations:

$$\mathbf{E}\left[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1\,\right] = \sum_{i=1}^{m} \mathbf{E}\left[\,Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1\,\right]$$

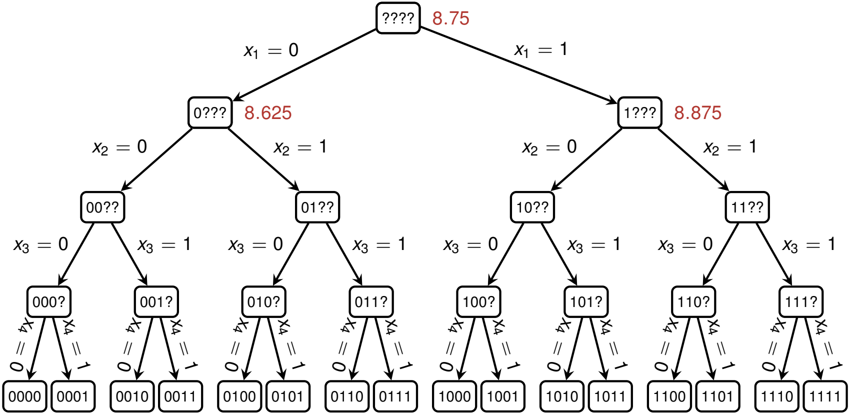
computable in $O(1)$

- **Step 2:** satisfies at least $7/8 \cdot m$ clauses
  - Due to the greedy choice in each iteration $j = 1, 2, \ldots, n$,

$$\begin{aligned}\mathbf{E}\left[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j\,\right] &\geq \mathbf{E}\left[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}\,\right] \\ &\geq \mathbf{E}\left[\,Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2}\,\right] \\ &\;\vdots \\ &\geq \mathbf{E}\left[\,Y\,\right] = \frac{7}{8} \cdot m. \qquad \square\end{aligned}$$

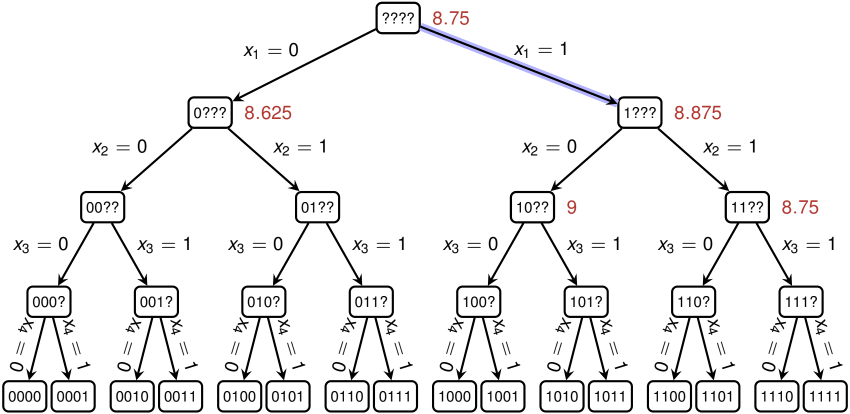
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge$
$(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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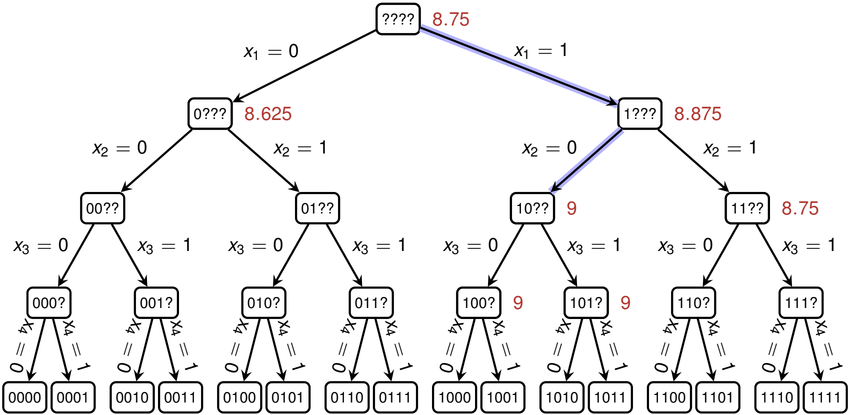
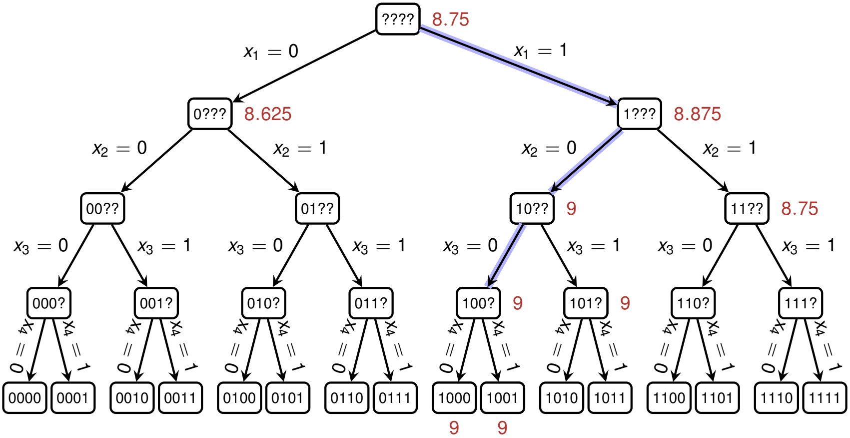


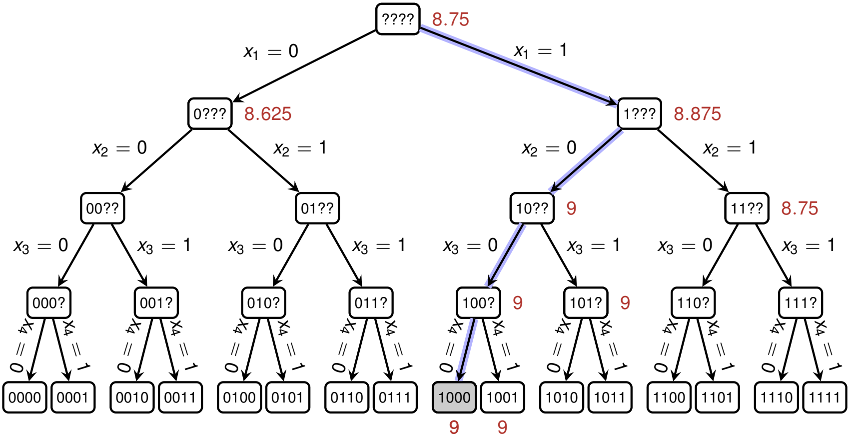
Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$$



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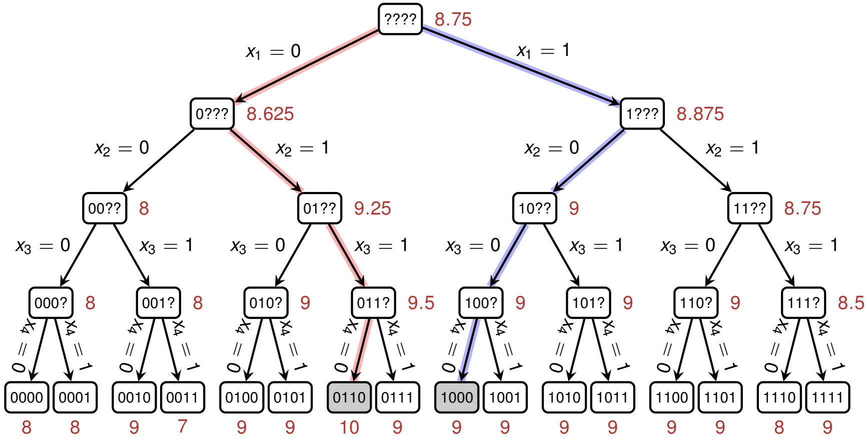


Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-SAT unless P=NP.

Essentially there is nothing smarter than just guessing!

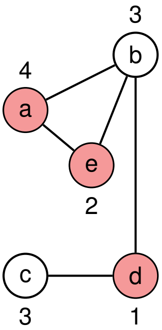
Outline

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.

This is (still) an NP-hard problem.



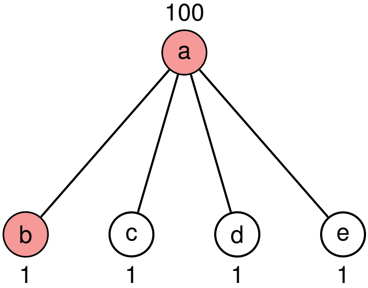
Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

The Greedy Approach from (Unweighted) Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
```



Computed solution has weight 101

The Greedy Approach from (Unweighted) Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E' = G.E
while E' ≠ ∅
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    C = C ∪ {u, v}
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return C
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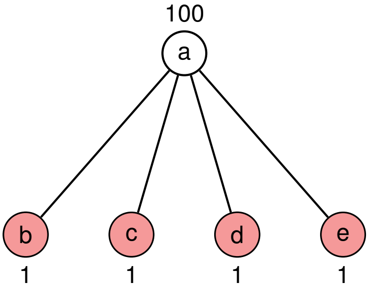


Optimal solution has weight 4

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

$$\begin{array}{llcll} \text{minimize} & \sum_{v \in V} w(v)x(v) & & & \\ \text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u,v) \in E \\ & x(v) & \in & \{0,1\} & \text{for each } v \in V \end{array}$$

optimum is a lower bound on the optimal weight of a minimum weight-cover.

Linear Program

$$\begin{array}{llcll} \text{minimize} & \sum_{v \in V} w(v)x(v) & & & \\ \text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u,v) \in E \\ & x(v) & \in & [0,1] & \text{for each } v \in V \end{array}$$

Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

```
C = ∅
compute x̄, an optimal solution to the linear program
for each v ∈ V
    if x̄(v) ≥ 1/2
        C = C ∪ {v}
return C
```

Theorem 35.7

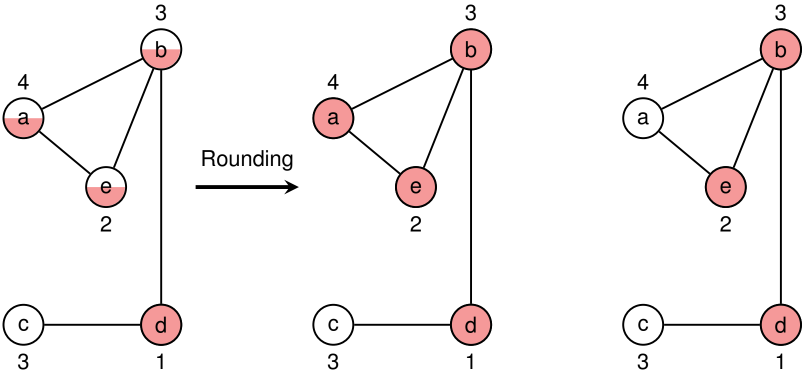
APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of Approx-Min-Weight-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}, \overline{x}(d) = 1, \overline{x}(c) = 0$

$x(a) = x(b) = x(e) = 1, x(d) = 1, x(c) = 0$



fractional solution of LP with weight $= 5.5$

rounded solution of LP with weight $= 10$

optimal solution with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2):

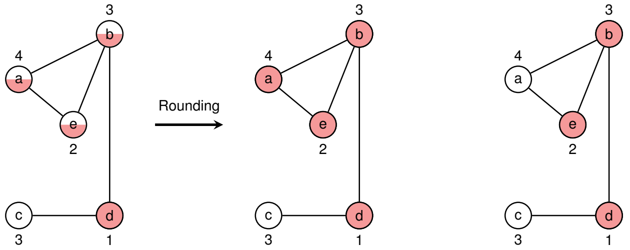
- Let C^* be an optimal solution to the minimum-weight vertex cover problem
- Let z^* be the value of an optimal solution to the linear program, so

$$z^* \leq w(C^*)$$

- **Step 1:** The computed set C covers all vertices:
  - Consider any edge $(u,v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$

  $\Rightarrow$ at least one of $\overline{x}(u)$ and $\overline{x}(v)$ is at least $1/2 \Rightarrow C$ covers edge (u,v)
- **Step 2:** The computed set C satisfies $w(C) \leq 2z^*$:

$$w(C^*) \geq z^* = \sum_{v \in V} w(v)\overline{x}(v) \geq \sum_{v \in V:\ \overline{x}(v) \geq 1/2} w(v) \cdot \frac{1}{2} = \frac{1}{2} w(C). \quad \square$$



Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

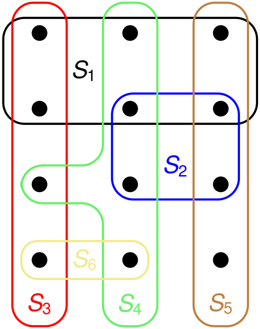
The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

Sum over the costs of all sets in $\mathcal{C}$



	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2

Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

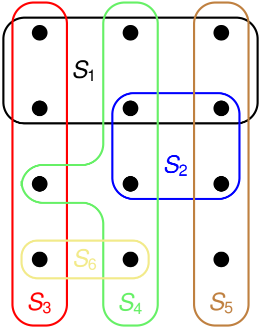
0-1 Integer Program

$$\begin{array}{llcll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\ \text{subject to} & \displaystyle\sum_{S \in \mathcal{F} :\, x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\ & y(S) & \in & \{0, 1\} & \text{for each } S \in \mathcal{F} \end{array}$$

Linear Program

$$\begin{array}{llcll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\ \text{subject to} & \displaystyle\sum_{S \in \mathcal{F} :\, x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\ & y(S) & \in & [0, 1] & \text{for each } S \in \mathcal{F} \end{array}$$

Back to the Example



	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2
$y(.)$:	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$

Cost equals 8.5

The strategy employed for Vertex-Cover would take all 6 sets!

Even worse: If all y's were below $1/2$, we would not even return a valid cover!

Randomised Rounding

	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2
$y(.)$:	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$

Idea: Interpret the y-values as probabilities for picking the respective set.

Randomised Rounding

- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $y(S)$.
- More precisely, if y denotes the optimal solution of the LP, then we compute an integral solution $\bar{y}$ by:

$$\bar{y}(S) = \begin{cases} 1 & \text{with probability } y(S) \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } S \in \mathcal{F}.$$

- Therefore, $\mathbf{E}[\,\bar{y}(S)\,] = y(S)$.

Randomised Rounding

	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2
$y(.)$:	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$

Idea: Interpret the y-values as probabilities for picking the respective set.

Lemma

- The expected cost satisfies

$$\mathbf{E}\left[\, c(\mathcal{C}) \,\right] = \sum_{S \in \mathcal{F}} c(S) \cdot y(S)$$

- The probability that an element $x \in X$ is covered satisfies

$$\mathbf{Pr}\left[x \in \bigcup_{S \in \mathcal{C}} S \right] \geq 1 - \frac{1}{e}.$$

Proof of Lemma

Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $y(S)$.

- The expected cost satisfies $\mathbf{E}[\,c(\mathcal{C})\,] = \sum_{S \in \mathcal{F}} c(S) \cdot y(S)$.
- The probability that x is covered satisfies $\mathbf{Pr}[\,x \in \cup_{S \in \mathcal{C}} S\,] \geq 1 - \frac{1}{e}$.

Proof:

- **Step 1**: The expected cost of the random set $\mathcal{C}$

$$\mathbf{E}[\,c(\mathcal{C})\,] = \mathbf{E}\left[\sum_{S \in \mathcal{C}} c(S)\right] = \mathbf{E}\left[\sum_{S \in \mathcal{F}} \mathbf{1}_{S \in \mathcal{C}} \cdot c(S)\right]$$

$$= \sum_{S \in \mathcal{F}} \mathbf{Pr}[\,S \in \mathcal{C}\,] \cdot c(S) = \sum_{S \in \mathcal{F}} y(S) \cdot c(S).$$

- **Step 2**: The probability for an element to be (not) covered

$$\mathbf{Pr}[\,x \notin \cup_{S \in \mathcal{C}} S\,] = \prod_{S \in \mathcal{F}\colon x \in S} \mathbf{Pr}[\,S \notin \mathcal{C}\,] = \prod_{S \in \mathcal{F}\colon x \in S} (1 - y(S))$$

$$\leq \prod_{S \in \mathcal{F}\colon x \in S} e^{-y(S)}$$

$$= e^{-\sum_{S \in \mathcal{F}\colon x \in S} y(S)} \leq e^{-1} \qquad \square$$

$1 + x \leq e^x$ for any $x \in \mathbb{R}$

y solves the LP!

The Final Step

Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $y(S)$.

- The expected cost satisfies $\mathbf{E}\left[\, c(\mathcal{C}) \,\right] = \sum_{S \in \mathcal{F}} c(S) \cdot y(S)$.
- The probability that x is covered satisfies $\mathbf{Pr}\left[\, x \in \cup_{S \in \mathcal{C}} S \,\right] \geq 1 - \frac{1}{e}$.

Problem: Need to make sure that every element is covered!

Idea: Amplify this probability by taking the union of $\Omega(\log n)$ random sets $\mathcal{C}$.

WEIGHTED SET COVER-LP$(X, \mathcal{F}, c)$

compute y, an optimal solution to the linear program
$\mathcal{C} = \emptyset$
repeat $2 \ln n$ times
  for each $S \in \mathcal{F}$
    let $\mathcal{C} = \mathcal{C} \cup \{S\}$ with probability $y(S)$
return $\mathcal{C}$

clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
- The expected approximation ratio is $2\ln(n)$.

Proof:

- **Step 1**: The probability that $\mathcal{C}$ is a cover
  - By previous Lemma, an element $x \in X$ is covered in one of the $2\ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

$$\mathbf{Pr}\left[\, x \notin \cup_{S \in \mathcal{C}} S \,\right] \leq \left(\frac{1}{e}\right)^{2\ln n} = \frac{1}{n^2}.$$

  - This implies for the event that all elements are covered:

$$\mathbf{Pr}\left[\, X = \cup_{S \in \mathcal{C}} S \,\right] = 1 - \mathbf{Pr}\left[\bigcup_{x \in X} \{x \notin \cup_{S \in \mathcal{C}} S\}\right]$$

$\mathbf{Pr}\left[\, A \cup B \,\right] \leq \mathbf{Pr}\left[\, A \,\right] + \mathbf{Pr}\left[\, B \,\right]$

$$\geq 1 - \sum_{x \in X} \mathbf{Pr}\left[\, x \notin \cup_{S \in \mathcal{C}} S \,\right] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.$$

- **Step 2**: The expected approximation ratio
  - By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot y(S)$.
  - Linearity $\Rightarrow$ $\mathbf{E}\left[\, c(\mathcal{C}) \,\right] \leq 2\ln(n) \cdot \sum_{S \in \mathcal{F}} c(S) \cdot y(S) \leq 2\ln(n) \cdot c(\mathcal{C}^*)$ $\square$

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
- The expected approximation ratio is $2\ln(n)$.

By Markov's inequality, $\mathbf{Pr}\left[\, c(\mathcal{C}) \leq 4\ln(n) \cdot c(\mathcal{C}^*) \,\right] \geq 1/2$.

Hence with probability at least $1 - \frac{1}{n} - \frac{1}{2} > \frac{1}{3}$, solution is within a factor of $4\ln(n)$ of the optimum.

probability could be further increased by repeating

Typical Approach for Designing Approximation Algorithms based on LPs

Spectrum of Approximations