Mini-ML expressions

M	$::=$	x	variable
	$\mid$	$\mathtt{true}$	boolean values
	$\mid$	$\mathtt{false}$	
	$\mid$	$\mathtt{if}\ M\ \mathtt{then}\ M\ \mathtt{else}\ M$	conditional
	$\mid$	$\lambda x\,(M)$	function abstraction
	$\mid$	$M\,M$	function application
	$\mid$	$\mathtt{let}\ x = M\ \mathtt{in}\ M$	local declaration
	$\mid$	$\mathtt{nil}$	nil list
	$\mid$	$M :: M$	list cons
	$\mid$	$\mathtt{case}\ M\ \mathtt{of}\ \mathtt{nil} \Rightarrow M \mid x :: x \Rightarrow M$	case expression

(abstract syntax trees)

Mini-ML types and type schemes

Types $\tau ::= \alpha$ type variable

$\mid$ $bool$ type of booleans

$\mid$ $\tau \to \tau$ function type

$\mid$ $\tau\ list$ list type

Mini-ML types and type schemes

Types

$$\begin{array}{rcll} \tau & ::= & \alpha & \text{type variable} \\ & \mid & \mathit{bool} & \text{type of booleans} \\ & \mid & \tau \to \tau & \text{function type} \\ & \mid & \tau\, \mathit{list} & \text{list type} \end{array}$$

where α ranges over a fixed, countably infinite set $\mathbf{TyVar}$.

Mini-ML types and type schemes

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Type Schemes

$$\sigma \quad ::= \quad \forall A\,(\tau)$$

where A ranges over finite subsets of the set $\mathbf{TyVar}$.

Mini-ML types and type schemes

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When $A = \{\alpha_1, \ldots, \alpha_n\}$ (mutually distinct type variables) we write $\forall A\,(\tau)$ as

$$\forall \alpha_1, \ldots, \alpha_n\,(\tau).$$

E.g.s of type schemes:

$\forall \alpha, \beta\,(\alpha \to \beta)$ $\quad \forall \alpha\,(\alpha\, list \to \beta)$

Mini-ML types and type schemes

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possibly empty, e.g. ↓

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E.g.s of type schemes:

$\forall \alpha, \beta\,(\alpha \to \beta) \quad \forall \alpha\,(\alpha\ list \to \beta) \quad \forall \{\,\}(\alpha \to bool)$

Mini-ML types and type schemes

Types

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When $A = \{\alpha_1, \ldots, \alpha_n\}$ (mutually distinct type variables) we write $\forall A\,(\tau)$ as

$$\forall \alpha_1, \ldots, \alpha_n\,(\tau).$$

When $A = \{\}$ is empty, we write $\forall A\,(\tau)$ just as τ. In other words, **we regard the set of types as a subset of the set of type schemes by identifying the type τ with the type scheme $\forall \{\,\}\,(\tau)$.**

Mini-ML typing judgement

takes the form

$$\Gamma \vdash M : \tau$$

where

- the *typing environment* Γ is a finite function from variables to *type schemes*.
  (We write $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ to indicate that Γ has domain of definition $dom(\Gamma) = \{x_1, \ldots, x_n\}$ (mutually distinct variables) and maps each x_i to the type scheme σ_i for $i = 1 \ldots n$.)
- M is a Mini-ML expression
- τ is a Mini-ML type.

Mini-ML type system, I

$$(\mathbf{var} \succ)\ \frac{}{\Gamma \vdash x : \tau} \text{ if } (x : \sigma) \in \Gamma \text{ and } \sigma \succ \tau$$

$$(\mathbf{bool})\ \frac{}{\Gamma \vdash B : bool} \text{ if } B \in \{\mathtt{true}, \mathtt{false}\}$$

$$(\mathbf{if})\ \frac{\Gamma \vdash M_1 : bool \qquad \Gamma \vdash M_2 : \tau \qquad \Gamma \vdash M_3 : \tau}{\Gamma \vdash (\mathtt{if}\ M_1\ \mathtt{then}\ M_2\ \mathtt{else}\ M_3) : \tau}$$

Specialising type schemes to types

A type τ is a *specialisation* of a type scheme $\sigma = \forall \alpha_1, \ldots, \alpha_n\, (\tau')$ if τ can be obtained from the type τ' by simultaneously substituting some types τ_i for the type variables α_i ($i = 1, \ldots, n$):

$$\tau = \tau'[\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n]$$

In this case we write $\boxed{\sigma \succ \tau}$

E.g. $\forall \alpha, \beta\, (\alpha \to \beta) \succ \beta \to bool$

& $\forall \alpha\, (\alpha \to \beta) \succ bool \to \beta$

but $\forall \alpha\, (\alpha \to \beta) \not\succ bool \to bool$

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(N.B. The relation is unaffected by the particular choice of names of bound type variables in σ.)

Identify type schemes up to renaming $\forall$-bound type variables, e.g.

$$\forall \alpha\,(\alpha \to \beta) = \forall \gamma\,(\gamma \to \beta) \neq \forall \gamma\,(\gamma \to \gamma)$$

Specialising type schemes to types

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(N.B. The relation is unaffected by the particular choice of names of bound type variables in σ.)

The converse relation is called *generalisation*: a type scheme σ generalises a type τ if $\sigma \succ \tau$.

Mini-ML type system, III

$$(\textbf{fn})\ \frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x\,(M) : \tau_1 \to \tau_2}\ \text{if } x \notin dom(\Gamma)$$

$$(\textbf{app})\ \frac{\Gamma \vdash M : \tau_1 \to \tau_2 \qquad \Gamma \vdash N : \tau_1}{\Gamma \vdash M\,N : \tau_2}$$

$\Gamma, x : \sigma$

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abbreviation for $x : \forall \{\}\ \tau_1$

Mini-ML type system, II

$$(\textbf{nil})\ \frac{}{\Gamma \vdash \texttt{nil} : \tau\ \mathit{list}}$$

$$(\textbf{cons})\ \frac{\Gamma \vdash M : \tau \qquad \Gamma \vdash L : \tau\ \mathit{list}}{\Gamma \vdash M :: L : \tau\ \mathit{list}}$$

$$(\textbf{case})\ \frac{\Gamma \vdash L : \tau\ \mathit{list} \qquad \Gamma \vdash N : \tau' \qquad \Gamma, x : \tau, \ell : \tau\ \mathit{list} \vdash C : \tau'}{\Gamma \vdash (\texttt{case}\ L\ \texttt{of}\ \texttt{nil} \Rightarrow N \mid x :: \ell \Rightarrow C) : \tau'}$$ if $x \neq \ell$ and $x, \ell \notin \mathit{dom}(\Gamma)$

e.g. $\{\} \vdash (\text{case true} :: \text{nil of nil} \Rightarrow \text{nil} \mid x :: \ell \Rightarrow \ell) : \text{bool list}$
can be deduced from the rules so far

Mini-ML type system, III

$$(\textbf{let})\ \frac{\begin{array}{c}\Gamma \vdash M_1 : \tau \\ \Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'\end{array}}{\Gamma \vdash (\mathtt{let}\, x = M_1 \,\mathtt{in}\, M_2) : \tau'}\ \text{if } x \notin dom(\Gamma) \text{ and } A = ftv(\tau) - ftv(\Gamma)$$

Mini-ML type system, III

$$\text{(let)}\ \frac{\Gamma \vdash M_1 : \tau \qquad \Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'}{\Gamma \vdash (\texttt{let}\ x = M_1\ \texttt{in}\ M_2) : \tau'}\ \text{if } x \notin dom(\Gamma) \text{ and } A = ftv(\tau) - ftv(\Gamma)$$

$ftv(\tau)$ = all type variables occuring in τ

$ftv\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\} = ftv(\sigma_1) \cup \cdots \cup ftv(\sigma_n)$
where if $\sigma = \forall A(\tau)$, then $ftv(\sigma) = ftv(\tau) - A$

Example of using the (**let**) rule

$$(\mathbf{let})\ \frac{\Gamma \vdash M_1 : \tau \qquad \Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'}{\Gamma \vdash (\texttt{let}\, x = M_1 \,\texttt{in}\, M_2) : \tau'}$$ if $x \notin dom(\Gamma)$ and $A = ftv(\tau) - ftv(\Gamma)$

If $\Gamma \vdash M_1 : \tau$ is $\boxed{y : \beta, z : \forall \gamma\,(\gamma \to \gamma \to bool) \vdash \lambda u\,(y) : \alpha \to \beta}$

Example of using the (let) rule

$$(\mathbf{let})\ \frac{\begin{array}{c}\Gamma \vdash M_1 : \tau \\ \Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'\end{array}}{\Gamma \vdash (\mathtt{let}\, x = M_1\, \mathtt{in}\, M_2) : \tau'} \text{ if } x \notin dom(\Gamma) \text{ and } A = ftv(\tau) - ftv(\Gamma)$$

If $\Gamma \vdash M_1 : \tau$ is $\boxed{y : \beta, z : \forall \gamma\,(\gamma \to \gamma \to bool) \vdash \lambda u\,(y) : \alpha \to \beta}$

then $A = \{\alpha, \beta\} - \{\beta\} = \{\alpha\}$ and $\forall A\,(\tau) = \forall \alpha\,(\alpha \to \beta)$.

Example of using the (**let**) rule

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So if $\Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'$ is

$$\boxed{y : \beta, z : \forall \gamma\,(\gamma \to \gamma \to bool), x : \forall \alpha\,(\alpha \to \beta) \vdash z\,(x\,y)\,(x\,\mathtt{nil}) : bool}$$

Example of using the (let) rule

$$(\mathbf{let})\ \frac{\Gamma \vdash M_1 : \tau \qquad \Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'}{\Gamma \vdash (\mathtt{let}\, x = M_1\, \mathtt{in}\, M_2) : \tau'}\ \text{if } x \notin dom(\Gamma) \text{ and } A = ftv(\tau) - ftv(\Gamma)$$

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then applying $(\mathbf{let})$ yields

$$\boxed{y : \beta, z : \forall\gamma\,(\gamma \to \gamma \to bool) \vdash \mathtt{let}\, x = \lambda u\,(y)\, \mathtt{in}\, z\,(x\,y)\,(x\,\mathtt{nil}) : bool}$$

Mini-ML type system, III

Definition. We write $\boxed{\Gamma \vdash M : \forall A\,(\tau)}$ to mean $\Gamma \vdash M : \tau$ is derivable from the Mini-ML typing rules and that $A = ftv(\tau) - ftv(\Gamma)$.

Mini-ML type system, III

$$(\mathbf{let})\ \frac{\begin{array}{c}\Gamma \vdash M_1 : \tau \\ \Gamma, x : \forall A\,(\tau) \vdash M_2 : \tau'\end{array}}{\Gamma \vdash (\mathtt{let}\,x = M_1\,\mathtt{in}\,M_2) : \tau'}\ \text{if } x \notin dom(\Gamma) \text{ and } A = ftv(\tau) - ftv(\Gamma)$$

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(So $(\mathbf{let})$ is equivalent to $\dfrac{\Gamma \vdash M_1 : \sigma \qquad \Gamma, x : \sigma \vdash M_2 : \tau'}{\Gamma \vdash (\mathtt{let}\,x = M_1\,\mathtt{in}\,M_2) : \tau'}$ if $x \notin dom(\Gamma)$.)

(Cf. Slide 6)

Mini-ML type-checking problem:

a given Γ, M & σ, does $\Gamma \vdash M : \sigma$ hold?

Mini-ML type-inference problem:

b given Γ & M, does there exist σ such ~~that~~ $\Gamma \vdash M : \sigma$ holds?

Solving a entails solving b, because of the form of the (let) typing rule.