Topics in Concurrency

Lecture 7

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The modal μ-calculus [§4.2 p48]

$$A ::= T \mid F \mid A_0 \wedge A_1 \mid A_0 \vee A_1 \mid \neg A \mid \langle \lambda \rangle A \mid \langle - \rangle A \mid X \mid \nu X.A$$

To guarantee monotonicity (and therefore the existence of the fixed point), require the variable X to occur only positively in A in $\nu X.A$. That is, X occurs only under an even number of $\neg$s.

$s \vDash \nu X.A$	iff	$s \in \nu X.A$
	i.e.	$s \in \bigcup \{S \subseteq \mathcal{P} \mid S \subseteq A[S/X]\}$
		the maximum fixed point of the monotonic function $S \mapsto A[S/X]$

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$$\begin{array}{lll} s \vDash \nu X.A & \text{iff} & s \in \nu X.A \\ & \text{i.e.} & s \in \bigcup\{S \subseteq \mathcal{P} \mid S \subseteq A[S/X]\} \\ & & \text{the maximum fixed point of the monotonic} \\ & & \text{function } S \mapsto A[S/X] \end{array}$$

As before, we take

$$[\lambda]A \equiv \neg\langle\lambda\rangle\neg A \qquad [-]A \equiv \neg\langle -\rangle\neg A$$

Now also take

$$\mu X.A \equiv \neg\nu X.(\neg A[\neg X/X])$$

Example

Consider the process

$$P \stackrel{\text{def}}{=} a.(a.P + b.c.\mathbf{nil})$$

Which states satisfy

- $\mu X.\langle a \rangle X$
- $\nu X.\langle a \rangle X$
- $\mu X.[a]X$
- $\nu X[a]X$

Approximants

Let $\varphi : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ be monotonic.
φ is $\bigcap$-continuous iff for all decreasing chains $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$

$$\bigcap_{n \in \omega} \varphi(X_n) = \varphi\left(\bigcap_{n \in \omega} X_n\right)$$

If the set of states $\mathcal{S}$ is finite, continuity certainly holds

Theorem

If $\varphi : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is $\bigcap$-continuous:

$$\nu X.\varphi(X) = \bigcap_{n \in \omega} \varphi^n(\mathcal{S})$$

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$$\bigcup_{n \in \omega} \varphi(X_n) = \varphi\left(\bigcup_{n \in \omega} X_n\right)$$

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If $\varphi : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is $\bigcup$-continuous:

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Proving interpretations

Proposition

$s \vDash \mu X.\langle a \rangle T \vee \langle - \rangle X$ in any transition system iff there exists a sequence of transitions from s to a state t where an a-action can occur.

Proving interpretations

Proposition

$s \models \nu X.\langle a \rangle X$ in a finite-state transition system iff there exists an infinite sequence of a-transitions from s.

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There are infinite-state transition systems where $\varphi(X) = \langle a \rangle X$ is not $\bigcap$-continuous.

Bisimilarity and modal μ

For finite-state processes, modal-μ can be encoded in infinitary H-M logic

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Note that logical equivalence in modal-μ does not generally imply bisimilarity (due to the lack of infinitary conjunction)