**Theorem**

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \quad (= \exp(x))$$

**Proof.**
If $x = 0$ then the result clearly holds and if $x \neq 0$ then

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\exp\left(n\ln\left(1+\frac{x}{n}\right)\right) = \lim_{n\to\infty}\exp\left(x\left(\frac{\ln(1+x/n)}{x/n}\right)\right) \\
&= \lim_{h\to 0}\exp\left(x\left(\frac{\ln(1+h)}{h}\right)\right) \\
&= \exp\left(x\left(\lim_{h\to 0}\frac{\ln(1+h)}{h}\right)\right) \\
&= \exp(x)
\end{aligned}$$

using the continuity of the $\exp()$ function and since $e^0 = 1$ so $\ln(1) = 0$ we have that

$$\lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \left.\frac{d(\ln(x))}{dx}\right|_{x=1} = \left.\left(\frac{1}{x}\right)\right|_{x=1} = 1\,.$$

□