L11: Algebraic Path Problems with applications to Internet Routing
Lectures 1, 2, and 3

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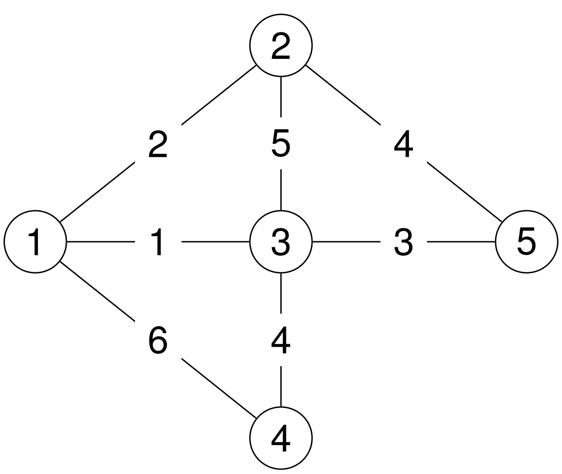
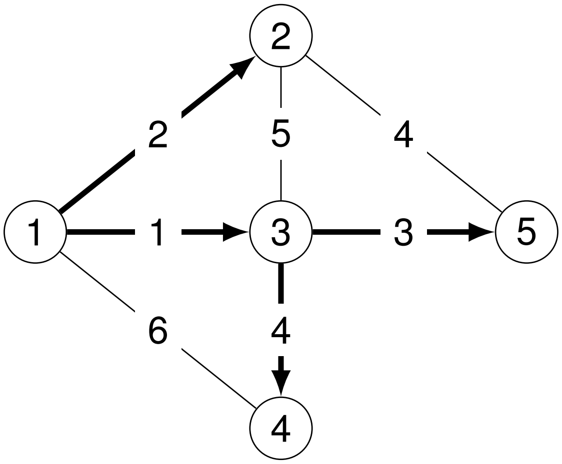
Shortest paths example, $\text{sp} = (\mathbb{N}^{\infty},\ \min,\ +, \infty,\ 0)$



The adjacency matrix

$$\mathbf{A} \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{array} \right] \end{array}$$

Shortest paths solution



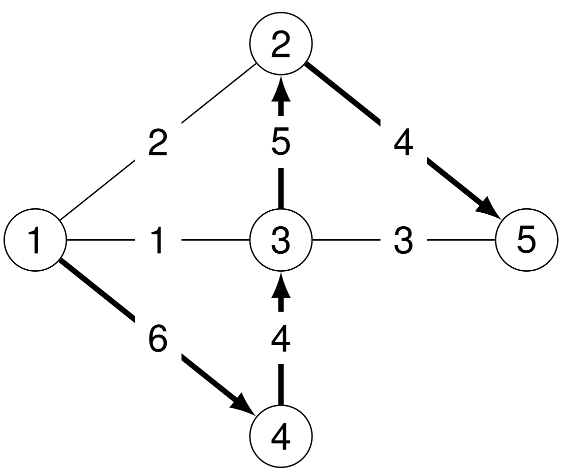
$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Widest paths example, $\mathrm{bw} = (\mathbb{N}^{\infty},\ \max,\ \min,\ 0,\ \infty)$



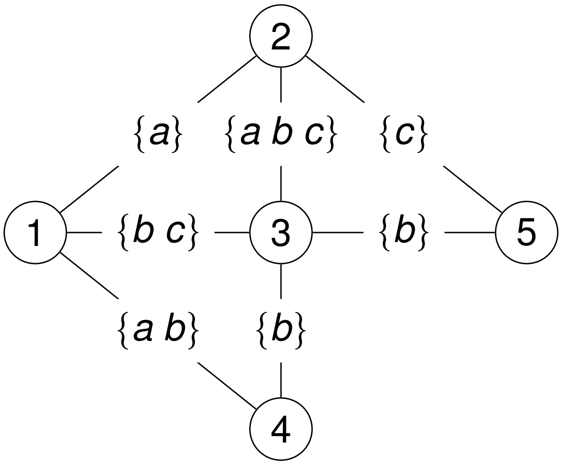
$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{array}\right] \end{array}$$

solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \max_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap, \{\}, \ \{a,\ b,\ c\})$



We want $\mathbf{A}^*$ to solve this global optimality problem:

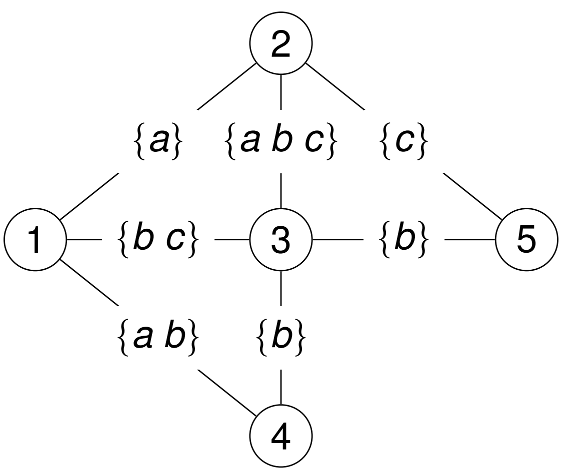
$$\mathbf{A}^*(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{A}^*(i,\ j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

$$\mathbf{A}^*(4,\ 1) = \{a,\ b\} \qquad \mathbf{A}^*(4,\ 5) = \{b\}$$

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$



We want matrix $\mathbf{R}$ to solve this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \bigcap_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the union of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that every path from i to j has at least one arc with weight containing x.

$$\mathbf{A}^*(4,\ 1) = \{b\} \qquad \mathbf{A}^*(4,\ 5) = \{b\} \qquad \mathbf{A}^*(5,\ 1) = \{\}$$

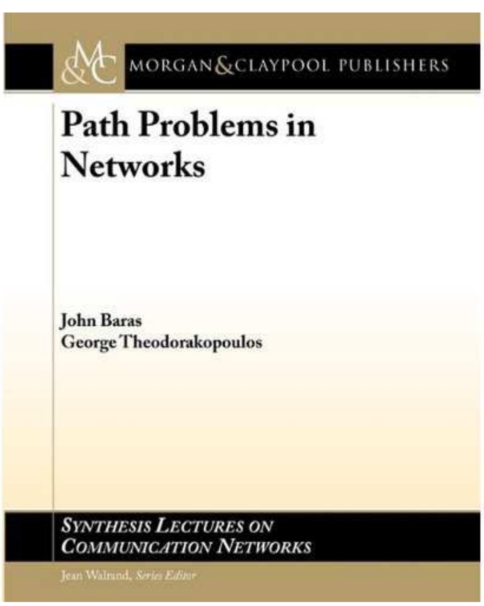
We will start by looking at Semirings

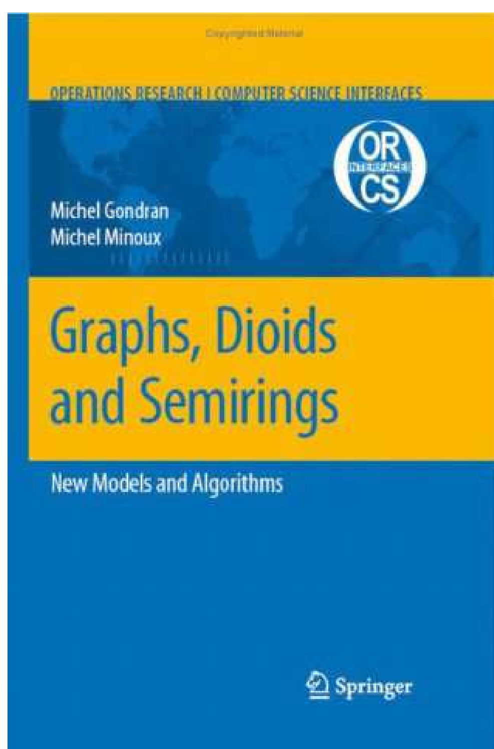
name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

A wee bit of notation!

Symbol	Interpretation
$\mathbb{N}$	Natural numbers (starting with zero)
$\mathbb{N}^{\infty}$	Natural numbers, plus infinity
$\overline{0}$	Identity for $\oplus$
$\overline{1}$	Identity for $\otimes$

Recommended Reading on Semiring Theory





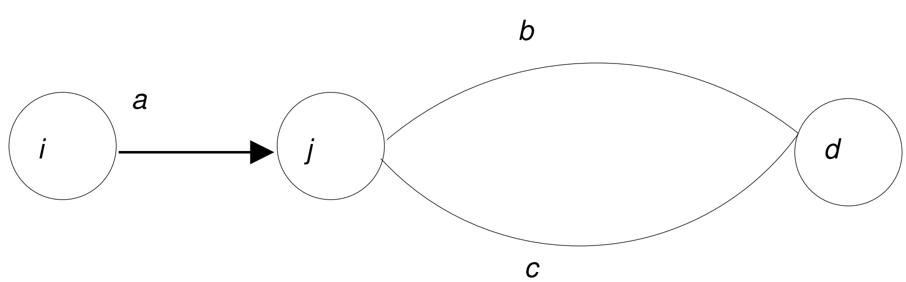
Semirings (generalise $(\mathbb{R}, +, \times, 0, 1)$)

We will look at the axioms of semirings. The most important are

distributivity

$$\begin{array}{lcrcl} \mathbb{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\ \mathbb{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c) \end{array}$$

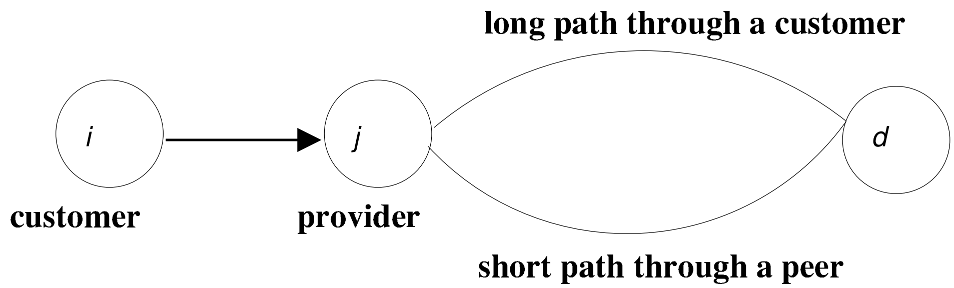
Distributivity, illustrated



$$a \otimes (b \oplus c) \quad = \quad (a \otimes b) \oplus (a \otimes c)$$

j makes the choice = i makes the choice

Should distributivity hold in Internet Routing? No!



- *j* prefers long path though one of its customers (not the shorter path through a competitor)
- given two routes from a provider, *i* prefers the one with a shorter path

More on this later in the term ...

The (**Tentative**) Plan

1	6 October	:	Motivation, overveiw
2	11 October	:	Semigroups
3	13 October	:	Semirgoups and partial orders
4	18 October	:	Semigroup Constructions
5	20 October	:	Semirings — Theory
6	25 October	:	Semirings — Constructions
7	27 October	:	Beyond Semirings — AMEs — “functions on arcs”
8	1 November	:	AME Constructions
9	3 November	:	Protocols : RIP, EIGRP (**HW 1 due noon 4 Nov**)
10	8 November	:	Inter-domain routing in the Internet I
11	10 November	:	Inter-domain routing in the Internet II
12	15 November	:	Beyond Semirings — Global vs Local optimality
13	17 November	:	More on Global vs Local optimality
14	22 November	:	Dijkstra revisited
15	24 November	:	Bellman-Ford revisited (**HW 2 due noon 25 Nov**)
16	29 November	:	Other algorithms
	17 January	:	**HW 3 due 17 Jan, 4pm**

Lectures 2, 3

- Semigroups
- A few important semigroup properties
- Semigroup and partial orders

Semigroups

Semigroup

A semigroup $(S, \bullet)$ is a non-empty set S with a binary operation such that

$$\mathbb{AS} \quad \text{associative} \quad \equiv \quad \forall a, b, c \in S, \ a \bullet (b \bullet c) = (a \bullet b) \bullet c$$

Important Assumption — We will ignore trival semigroups

We will impicitly assume that $2 \leqslant \mid S \mid$.

Note

Many useful binary operations are not semigroup operations. For example, $(\mathbb{R}, \bullet)$, where $a \bullet b \equiv (a + b)/2$.

Some Important Semigroup Properties

$\mathbb{ID}$	identity	$\equiv$	$\exists \alpha \in S,\ \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha$
$\mathbb{AN}$	annihilator	$\equiv$	$\exists \omega \in S,\ \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega$
$\mathbb{CM}$	commutative	$\equiv$	$\forall a, b \in S,\ a \bullet b = b \bullet a$
$\mathbb{SL}$	selective	$\equiv$	$\forall a, b \in S,\ a \bullet b \in \{a,\ b\}$
$\mathbb{IP}$	idempotent	$\equiv$	$\forall a \in S,\ a \bullet a = a$

A semigroup with an identity is called a monoid.

Note that

$$\mathbb{SL}(S,\ \bullet) \implies \mathbb{IP}(S,\ \bullet)$$

A few concrete semigroups

S	$\bullet$	description	α	ω	$\mathbb{CM}$	$\mathbb{SL}$	$\mathbb{IP}$
S	left	x left $y = x$				$\star$	$\star$
S	right	x right $y = y$				$\star$	$\star$
S^*	$\cdot$	concatenation	ϵ				
S^+	$\cdot$	concatenation					
$\{t,\ f\}$	$\wedge$	conjunction	t	f	$\star$	$\star$	$\star$
$\{t,\ f\}$	$\vee$	disjunction	f	t	$\star$	$\star$	$\star$
$\mathbb{N}$	min	minimum		0	$\star$	$\star$	$\star$
$\mathbb{N}$	max	maximum	0		$\star$	$\star$	$\star$
2^W	$\cup$	union	$\{\}$	W	$\star$		$\star$
2^W	$\cap$	intersection	W	$\{\}$	$\star$		$\star$
$\mathrm{fin}(2^U)$	$\cup$	union	$\{\}$		$\star$		$\star$
$\mathrm{fin}(2^U)$	$\cap$	intersection		$\{\}$	$\star$		$\star$
$\mathbb{N}$	$+$	addition	0		$\star$		
$\mathbb{N}$	$\times$	multiplication	1	0	$\star$		

W a finite set, U an infinite set. For set Y, $\mathrm{fin}(Y) \equiv \{X \in Y \mid X \text{ is finite}\}$

A few abstract semigroups

S	$\bullet$	description	α	ω	$\mathbb{CM}$	$\mathbb{SL}$	$\mathbb{IP}$
2^U	$\cup$	union	$\{\}$	U	$\star$		$\star$
2^U	$\cap$	intersection	U	$\{\}$	$\star$		$\star$
$2^{U\times U}$	$\bowtie$	relational join	$\mathcal{I}_U$	$\{\}$			
$X \to X$	$\circ$	composition	$\lambda x.x$				

U an infinite set
$X \bowtie Y \equiv \{(x, z) \in U \times U \mid \exists y \in U, \ (x, \ y) \in X \wedge (y, \ z) \in Y\}$
$\mathcal{I}_U \equiv \{(u, \ u) \mid u \in U\}$

subsemigroup

Suppose $(S, \ \bullet)$ is a semigroup and $T \subseteq S$. If T is closed w.r.t $\bullet$ (that is, $\forall x, y \in T, x \bullet y \in T$), then $(T, \ \bullet)$ is a subsemigroup of S.

Order Relations

We are interested in order relations $\leqslant \subseteq \ S \times S$

Definition (Important Order Properties)

$$\begin{array}{lrcl} \mathbb{RX} & \text{reflexive} & \equiv & a \leqslant a \\ \mathbb{TR} & \text{transitive} & \equiv & a \leqslant b \wedge b \leqslant c \to a \leqslant c \\ \mathbb{AY} & \text{antisymmetric} & \equiv & a \leqslant b \wedge b \leqslant a \to a = b \\ \mathbb{TO} & \text{total} & \equiv & a \leqslant b \vee b \leqslant a \end{array}$$

	pre-order	partial order	preference order	total order
$\mathbb{RX}$	$\star$	$\star$	$\star$	$\star$
$\mathbb{TR}$	$\star$	$\star$	$\star$	$\star$
$\mathbb{AY}$		$\star$		$\star$
$\mathbb{TO}$			$\star$	$\star$

Canonical Pre-order of a Commutative Semigroup

Definition (Canonical pre-orders)

$$a \trianglelefteq_{\bullet}^{R} b \equiv \exists c \in S : b = a \bullet c$$
$$a \trianglelefteq_{\bullet}^{L} b \equiv \exists c \in S : a = b \bullet c$$

Lemma (Sanity check)

Associativity of $\bullet$ implies that these relations are transitive.

Proof.

Note that $a \trianglelefteq_{\bullet}^{R} b$ means $\exists c_1 \in S : b = a \bullet c_1$, and $b \trianglelefteq_{\bullet}^{R} c$ means $\exists c_2 \in S : c = b \bullet c_2$. Letting $c_3 = c_1 \bullet c_2$ we have $c = b \bullet c_2 = (a \bullet c_1) \bullet c_2 = a \bullet (c_1 \bullet c_2) = a \bullet c_3$. That is, $\exists c_3 \in S : c = a \bullet c_3$, so $a \trianglelefteq_{\bullet}^{R} c$. The proof for $\trianglelefteq_{\bullet}^{L}$ is similar. □

Canonically Ordered Semigroup

Definition (Canonically Ordered Semigroup)

A commutative semigroup $(S, \bullet)$ is canonically ordered when $a \trianglelefteq_{\bullet}^{R} c$ and $a \trianglelefteq_{\bullet}^{L} c$ are partial orders.

Definition (Groups)

A monoid is a group if for every $a \in S$ there exists a $a^{-1} \in S$ such that $a \bullet a^{-1} = a^{-1} \bullet a = \alpha$.

Canonically Ordered Semigroups vs. Groups

Lemma (THE BIG DIVIDE)

Only a trivial group is canonically ordered.

Proof.

If $a, \ b \in S$, then $a = \alpha_{\bullet} \bullet a = (b \bullet b^{-1}) \bullet a = b \bullet (b^{-1} \bullet a) = b \bullet c$, for $c = b^{-1} \bullet a$, so $a \trianglelefteq^{L}_{\bullet} b$. In a similar way, $b \trianglelefteq^{R}_{\bullet} a$. Therefore $a = b$. □

Natural Orders

Definition (Natural orders)

Let $(S, \ \bullet)$ be a semigroup.

$$a \leqslant^{L}_{\bullet} b \equiv a = a \bullet b$$
$$a \leqslant^{R}_{\bullet} b \equiv b = a \bullet b$$

Lemma

If $\bullet$ is commutative and idempotent, then $a \trianglelefteq^{D}_{\bullet} b \iff a \leqslant^{D}_{\bullet} b$, for $D \in \{R, \ L\}$.

Proof.

$$\begin{array}{rcl} a \trianglelefteq^{R}_{\bullet} b & \iff & b = a \bullet c = (a \bullet a) \bullet c = a \bullet (a \bullet c) \\ & = & a \bullet b \iff a \leqslant^{R}_{\bullet} b \\ a \trianglelefteq^{L}_{\bullet} b & \iff & a = b \bullet c = (b \bullet b) \bullet c = b \bullet (b \bullet c) \\ & = & b \bullet a = a \bullet b \iff a \leqslant^{L}_{\bullet} b \end{array}$$

□

Special elements and natural orders

Lemma (Natural Bounds)

- *If α exists, then for all a, $a \leqslant_{\bullet}^{L} \alpha$ and $\alpha \leqslant_{\bullet}^{R} a$*
- *If ω exists, then for all a, $\omega \leqslant_{\bullet}^{L} a$ and $a \leqslant_{\bullet}^{R} \omega$*
- *If α and ω exist, then S is bounded.*

$$\begin{array}{ccccc} \omega & \leqslant_{\bullet}^{L} & a & \leqslant_{\bullet}^{L} & \alpha \\ \alpha & \leqslant_{\bullet}^{R} & a & \leqslant_{\bullet}^{R} & \omega \end{array}$$

Remark (Thanks to Iljitsch van Beijnum)

Note that this means for (min, $+$) we have

$$\begin{array}{ccccc} 0 & \leqslant_{\min}^{L} & a & \leqslant_{\min}^{L} & \infty \\ \infty & \leqslant_{\min}^{R} & a & \leqslant_{\min}^{R} & 0 \end{array}$$

and still say that this is bounded, even though one might argue with the terminology!

Examples of special elements

S	$\bullet$	α	ω	$\leqslant_{\bullet}^{L}$	$\leqslant_{\bullet}^{R}$
$\mathbb{N}^{\infty}$	min	∞	0	$\leqslant$	$\geqslant$
$\mathbb{N}^{-\infty}$	max	0	$-\infty$	$\geqslant$	$\leqslant$
$\mathcal{P}(W)$	$\cup$	$\{\}$	W	$\subseteq$	$\supseteq$
$\mathcal{P}(W)$	$\cap$	W	$\{\}$	$\supseteq$	$\subseteq$

Property Management

Lemma

Let $D \in \{R,\ L\}$.

1. $\mathbb{IP}(S,\ \bullet) \iff \mathbb{RX}(S,\ \leqslant^D_\bullet)$
2. $\mathbb{CM}(S,\ \bullet) \implies \mathbb{AY}(S,\ \leqslant^D_\bullet)$
3. $\mathbb{AS}(S,\ \bullet) \implies \mathbb{TR}(S,\ \leqslant^D_\bullet)$
4. $\mathbb{CM}(S,\ \bullet) \implies (\mathbb{SL}(S,\ \bullet) \iff \mathbb{TO}(S,\ \leqslant^D_\bullet))$

Proof.

1. $a \leqslant^D_\bullet a \iff a = a \bullet a$,
2. $a \leqslant^L_\bullet b \wedge b \leqslant^L_\bullet a \iff a = a \bullet b \wedge b = b \bullet a \implies a = b$
3. $a \leqslant^L_\bullet b \wedge b \leqslant^L_\bullet c \iff a = a \bullet b \wedge b = b \bullet c \implies a = a \bullet (b \bullet c) = (a \bullet b) \bullet c = a \bullet c \implies a \leqslant^L_\bullet c$
4. $a = a \bullet b \vee b = a \bullet b \iff a \leqslant^L_\bullet b \vee b \leqslant^L_\bullet a$

□

Bounds

Suppose $(S,\ \leqslant)$ is a partially ordered set.

greatest lower bound

For $a,\ b \in S$, the element $c \in S$ is the greatest lower bound of a and b, written $c = a \text{ glb } b$, if it is a lower bound ($c \leqslant a$ and $c \leqslant b$), and for every $d \in S$ with $d \leqslant a$ and $d \leqslant b$, we have $d \leqslant c$.

least upper bound

For $a,\ b \in S$, the element $c \in S$ is the least upper bound of a and b, written $c = a \text{ lub } b$, if it is an upper bound ($a \leqslant c$ and $b \leqslant c$), and for every $d \in S$ with $a \leqslant d$ and $b \leqslant d$, we have $c \leqslant d$.

Semi-lattices

Suppose $(S, \leqslant)$ is a partially ordered set.

meet-semilattice

S is a meet-semilattice if $a \ \mathrm{glb}\ b$ exists for each $a, b \in S$.

join-semilattice

S is a join-semilattice if $a \ \mathrm{lub}\ b$ exists for each $a, b \in S$.

Fun Facts

Fact 1

Suppose $(S, \bullet)$ is a commutative and idempotent semigroup.

- $(S, \leqslant^L_\bullet)$ is a meet-semilattice with $a \ \mathrm{glb}\ b = a \bullet b$.
- $(S, \leqslant^R_\bullet)$ is a join-semilattice with $a \ \mathrm{lub}\ b = a \bullet b$.

Fact 2

Suppose $(S, \leqslant)$ is a partially ordered set.

- If $(S, \leqslant)$ is a meet-semilattice, then (S, glb) is a commutative and idempotent semigroup.
- If $(S, \leqslant)$ is a join-semilattice, then (S, lub) is a commutative and idempotent semigroup.

That is, semi-lattices represent the same class of structures as commutative and idempotent semigroups.

Lectures 4, 5

- Semigroup Constructions
- Homework 1
- Semirings
- Matrix semirings
- Shortest paths

Add identity

$$\text{AddId}(\alpha,\ (S,\ \bullet)) \equiv (S \uplus \{\alpha\}, \bullet_{\alpha}^{\text{id}})$$

where

$$a \bullet_{\alpha}^{\text{id}} b \ \equiv \ \begin{cases} a & (\text{if } b = \text{inr}(\alpha)) \\ b & (\text{if } a = \text{inr}(\alpha)) \\ \text{inl}(x \bullet y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

disjoint union

$$A \uplus B \equiv \{\text{inl}(a) \mid a \in A\} \cup \{\text{inr}(b) \mid b \in B\}$$

Add identity

Easy Exercises

$$
\begin{array}{rcl}
\mathbb{AS}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{AN}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \\
\mathbb{CM}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{IP}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\
\mathbb{SL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet)
\end{array}
$$

Inserting an annihilator

$$\mathrm{AddAn}(\omega,\ (S,\ \bullet)) \equiv (S \uplus \{\omega\}, \bullet_{\omega}^{\mathrm{an}})$$

where

$$
a \bullet_{\omega}^{\mathrm{an}} b \equiv \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}
$$

Add annihilator

Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \\ \mathbb{AN}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{CM}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

Lexicographic Product of Semigroups

Lexicographic product semigroup

Suppose that semigroup $(S,\ \bullet)$ is commutative, idempotent, and selective and that $(T,\ \diamond)$ is a semigroup.

$$(S,\ \bullet)\ \vec{\times}\ (T,\ \diamond) \equiv (S \times T,\ \star)$$

where $\star \equiv \bullet\ \vec{\times}\ \diamond$ is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = \begin{cases} (s_1 \bullet s_2,\ t_1 \diamond t_2) & s_1 = s_1 \bullet s_2 = s_2 \\ (s_1 \bullet s_2,\ t_1) & s_1 = s_1 \bullet s_2 \neq s_2 \\ (s_1 \bullet s_2,\ t_2) & s_1 \neq s_1 \bullet s_2 = s_2 \end{cases}$$

Examples

$(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \min)$

$$\begin{array}{rcl}(1,\ 17) \star (2,3) & = & (1,17) \\ (2,\ 17) \star (2,3) & = & (2,3) \\ (2,\ 3) \star (2,3) & = & (2,3)\end{array}$$

$(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \max)$

$$\begin{array}{rcl}(1,\ 17) \star (2,3) & = & (1,17) \\ (2,\ 17) \star (2,3) & = & (2,17) \\ (2,\ 3) \star (2,3) & = & (2,3)\end{array}$$

$(\mathbb{N},\ \max)\ \vec{\times}\ (\mathbb{N},\ \min)$

$$\begin{array}{rcl}(1,\ 17) \star (2,3) & = & (2,3) \\ (2,\ 17) \star (2,3) & = & (2,3) \\ (2,\ 3) \star (2,3) & = & (2,3)\end{array}$$

Assuming $\mathbb{AS}(S, \bullet) \wedge \mathbb{CM}(S, \bullet) \wedge \mathbb{IP}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$

$$\begin{array}{rcl}\mathbb{AS}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{AS}(T, \diamond) \\ \mathbb{ID}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\ \mathbb{AN}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\ \mathbb{CM}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{CM}(T, \diamond) \\ \mathbb{IP}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{IP}(T, \diamond) \\ \mathbb{SL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{SL}(T, \diamond) \\ \mathbb{IR}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\ \mathbb{IL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE}\end{array}$$

All easy, except for $\mathbb{AS}$ (See Homework 1!).

Direct Product of Semigroups

Let $(S, \bullet)$ and $(T, \diamond)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted

$$(S, \bullet) \times (T, \diamond) \equiv (S \times T, \star)$$

where

$$\star = \bullet \times \diamond$$

is defined as

$$(s_1, t_1) \star (s_2, t_2) = (s_1 \bullet s_2, t_1 \diamond t_2).$$

Easy exercises

$$\begin{array}{rcl}
\mathbb{AS}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \wedge \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \wedge \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \wedge \mathbb{IP}(T, \diamond)
\end{array}$$

What about $\mathbb{SL}$?

Consider the product of two selective semigroups, such as $(\mathbb{N}, \min) \times (\mathbb{N}, \max)$.

$$(10, 10) \star (1, 3) = (1, 10) \notin \{(10, 10), (1, 3)\}$$

The result in this case is not selective!

Direct product and $\mathbb{SL}$?

$$\mathbb{SL}((S, \bullet) \times (T, \diamond)) \;\Leftrightarrow\; (\mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond)) \vee (\mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond))$$

$$\begin{array}{lrcl} \mathbb{IR} & \text{is right} & \equiv & \forall s, t \in S, s \bullet t = t \\ \mathbb{IL} & \text{is left} & \equiv & \forall s, t \in S, s \bullet t = s \end{array}$$

See Homework 1

$$\begin{array}{rcl} \mathbb{IR}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond) \\ \mathbb{IL}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond) \end{array}$$

Revisit other constructions ...

$$\begin{array}{rcl} \mathbb{IR}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\ \mathbb{IL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\ & & \\ \mathbb{IR}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\ \mathbb{IL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \end{array}$$

Assuming $\mathbb{AS}(S, \bullet) \wedge \mathbb{CM}(S, \bullet) \wedge \mathbb{IP}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$

$$\begin{array}{rcl} \mathbb{IR}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\ \mathbb{IL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \end{array}$$

Lifted Product

Lifted product semigroup

Assume $(S, \bullet)$ is a semigroup. Let $\text{lift}(S, \bullet) \equiv (\text{fin}(2^S), \hat{\bullet})$ where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X, y \in Y\}.$$

$$\{1, 3, 17\} \hat{+} \{1, 3, 17\} = \{2, 4, 6, 18, 20, 34\}$$

$$\begin{array}{rcl}
\mathbb{AS}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \ \ (\hat{\alpha} = \{\alpha\}) \\
\mathbb{AN}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{TRUE} \ \ (\omega = \{\}) \\
\mathbb{CM}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{SL}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \ \wedge \mid S \mid = 2) \\
\mathbb{IP}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{SL}((S, \bullet)) \\
\mathbb{IL}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}$$

Why bother with all of these $\Leftrightarrow$ rules?

I would rather calculate than prove!

$$
\begin{array}{ll}
 & \mathbb{IP}(\text{lift}(\text{lift}(\{t,\ f\},\ \wedge)) \\
\Leftrightarrow & \mathbb{SL}(\{t,\ f\},\ \wedge) \\
\Leftrightarrow & \mathbb{IL}(\{t,\ f\},\ \wedge) \vee \mathbb{IR}(\{t,\ f\},\ \wedge) \vee (\mathbb{IP}(\{t,\ f\},\ \wedge)\ \wedge \mid \{t,\ f\} \mid = 2) \\
\Leftrightarrow & \mathbb{FALSE} \vee \mathbb{FALSE} \vee (\mathbb{TRUE} \wedge \mathbb{TRUE}) \\
\Leftrightarrow & \mathbb{TRUE}
\end{array}
$$

Note

This kind of calculation will become more interesting as we introduce more complex constructors and consider more complex properties — such as those associated with semirings.

Homework 1

Each question is 25 points.

1. Prove Fact 1
2. Prove Fact 2
3. Prove

$$
\begin{array}{c}
\mathbb{SL}((S, \bullet) \times (T, \diamond)) \\
\Leftrightarrow \\
(\mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond)) \vee (\mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond))
\end{array}
$$

4. (Rather difficult). Prove

$$
\begin{array}{c}
\mathbb{SL}(\text{lift}(S, \bullet)) \\
\Leftrightarrow \\
\mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet)\ \wedge \mid S \mid = 2)
\end{array}
$$

Bi-semigroups and Pre-Semirings

$(S, \oplus, \otimes)$ is a bi-semigroup when

- $(S, \oplus)$ is a semigroup
- $(S, \otimes)$ is a semigroup

$(S, \oplus, \otimes)$ is a pre-semiring when

- $(S, \oplus, \otimes)$ is a bi-semigroup
- $\oplus$ is commutative

and left- and right-distributivity hold,

$$\begin{array}{lcrcl} \mathbb{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\ \mathbb{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c) \end{array}$$

Semirings

$(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring when

- $(S, \oplus, \otimes)$ is a pre-semiring
- $(S, \oplus, \overline{0})$ is a (commutative) monoid
- $(S, \otimes, \overline{1})$ is a monoid
- $\overline{0}$ is an annihilator for $\otimes$

Examples

Pre-semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
min_plus	$\mathbb{N}$	min	$+$		0
max_min	$\mathbb{N}$	max	min	0	

Semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0
bw	$\mathbb{N}^{\infty}$	max	min	0	∞

Note the sloppiness — the symbols $+$, max, and min in the two tables represent different functions....

How about (max, $+$)?

Pre-semiring

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
max_plus	$\mathbb{N}$	max	$+$	0	0

- What about "$\overline{0}$ is an annihilator for $\otimes$"? No!

Fix that ...

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
max_plus$^{-\infty}$	$\mathbb{N} \uplus \{-\infty\}$	max	$+$	$-\infty$	0

Matrix Semirings

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- Define the semiring of $n \times n$-matrices over S : $(\mathbb{M}_n(S), \oplus, \otimes, \mathbf{J}, \mathbf{I})$

$\oplus$ and $\otimes$

$$\begin{aligned}(\mathbf{A} \oplus \mathbf{B})(i, j) &= \mathbf{A}(i, j) \oplus \mathbf{B}(i, j) \\ (\mathbf{A} \otimes \mathbf{B})(i, j) &= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)\end{aligned}$$

J and I

$$\begin{aligned}\mathbf{J}(i, j) &= \overline{0} \\ \mathbf{I}(i, j) &= \begin{cases} \overline{1} & (\text{if } i = j) \\ \overline{0} & (\text{otherwise}) \end{cases}\end{aligned}$$

$\mathbb{M}_n(S)$ is a semiring!

For example, here is left distribution

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

$$\begin{aligned} & (\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}))(i, j) \\ = & \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes (\mathbf{B} \oplus \mathbf{C})(q, j) \\ = & \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes (\mathbf{B}(q, j) \oplus \mathbf{C}(q, j)) \\ = & \bigoplus_{1 \leqslant q \leqslant n} (\mathbf{A}(i, q) \otimes \mathbf{B}(q, j)) \oplus (\mathbf{A}(i, q) \otimes \mathbf{C}(q, j)) \\ = & (\bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)) \oplus (\bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \otimes \mathbf{C}(q, j)) \\ = & ((\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}))(i, j) \end{aligned}$$

Note : we only needed left-distributivity on S.

Matrix encoding path problems

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- $G = (V, E)$ a directed graph
- $w \in E \to S$ a weight function

Path weight

The weight of a path $p = i_1, i_2, i_3, \cdots, i_k$ is

$$w(p) = w(i_1, i_2) \otimes w(i_2, i_3) \otimes \cdots \otimes w(i_{k-1}, i_k).$$

The empty path is given the weight $\overline{1}$.

Adjacency matrix $\mathbf{A}$

$$\mathbf{A}(i, j) = \begin{cases} w(i, j) & \text{if } (i, j) \in E, \\ \overline{0} & \text{otherwise} \end{cases}$$

The general problem of finding globally optimal path weights

Given an adjacency matrix $\mathbf{A}$, find $\mathbf{A}^*$ such that for all $i, j \in V$

$$\mathbf{A}^*(i, j) = \bigoplus_{p \in P(i, j)} w(p)$$

where $P(i, j)$ represents the set of all paths from i to j.

How can we solve this problem?

Matrix methods

Matrix powers, $\mathbf{A}^k$

$$\mathbf{A}^0 = \mathbf{I}$$
$$\mathbf{A}^{k+1} = \mathbf{A} \otimes \mathbf{A}^k$$

Closure, $\mathbf{A}^*$

$$\mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$
$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots$$

Note: $\mathbf{A}^*$ might not exist. Why?

Matrix methods can compute optimal path weights

- Let $P(i, j)$ be the set of paths from i to j.
- Let $P^k(i, j)$ be the set of paths from i to j with exactly k arcs.
- Let $P^{(k)}(i, j)$ be the set of paths from i to j with at most k arcs.

Theorem

$$(1) \quad \mathbf{A}^k(i,\ j) = \bigoplus_{p \in P^k(i,\ j)} w(p)$$
$$(2) \quad \mathbf{A}^{(k)}(i,\ j) = \bigoplus_{p \in P^{(k)}(i,\ j)} w(p)$$
$$(3) \quad \mathbf{A}^*(i,\ j) = \bigoplus_{p \in P(i,\ j)} w(p)$$

Warning again: for some semirings the expression $\mathbf{A}^*(i,\ j)$ might not be well-defeind. Why?

Proof of (1)

By induction on k. Base Case: $k = 0$.

$$P^0(i,\ i) = \{\epsilon\},$$

so $\mathbf{A}^0(i\ ,i) = \mathbf{I}(i,\ i) = \overline{1} = w(\epsilon)$.

And $i \neq j$ implies $P^0(i,j) = \{\}$. By convention

$$\bigoplus_{p \in \{\}} w(p) = \overline{0} = \mathbf{I}(i,\ j).$$

Proof of (1)

Induction step.

$$\begin{aligned}
\mathbf{A}^{k+1}(i,j) &= (\mathbf{A} \otimes \mathbf{A}^k)(i,\ j) \\
&= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i,\ q) \otimes \mathbf{A}^k(q,\ j) \\
&= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i,\ q) \otimes (\bigoplus_{p \in P^k(q,\ j)} w(p)) \\
&= \bigoplus_{1 \leqslant q \leqslant n} \bigoplus_{p \in P^k(q,\ j)} \mathbf{A}(i,\ q) \otimes w(p) \\
&= \bigoplus_{(i,\ q) \in E} \bigoplus_{p \in P^k(q,j)} w(i,\ q) \otimes w(p) \\
&= \bigoplus_{p \in P^{k+1}(i,\ j)} w(p)
\end{aligned}$$

When does $\mathbf{A}^*$ exist? Try a general approach.

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring

Powers, a^k

$$\begin{array}{rcl} a^0 & = & \overline{1} \\ a^{k+1} & = & a \otimes a^k \end{array}$$

Closure, a^*

$$\begin{array}{rcl} a^{(k)} & = & a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\ a^* & = & a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots \end{array}$$

Definition (q stability)

If there exists a q such that $a^{(q)} = a^{(q+1)}$, then a is q-stable. By induction: $\forall t, 0 \leqslant t, a^{(q+t)} = a^{(q)}$. Therefore, $a^* = a^{(q)}$.

Fun Facts

Fact 3

If $\overline{1}$ is an annihiltor for $\oplus$, then every $a \in S$ is 0-stable!

Fact 4

If S is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

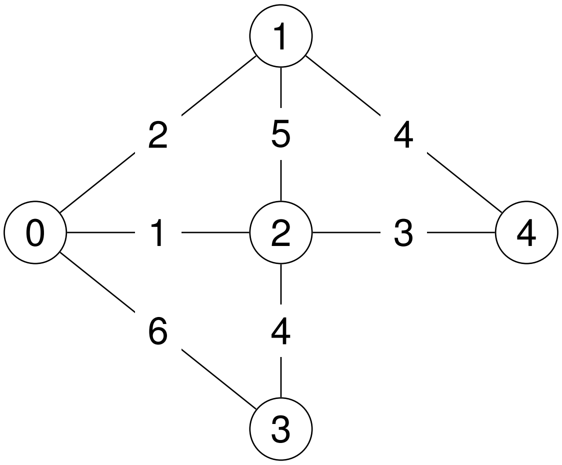
$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

Why? Because we can ignore paths with loops.

$$(a \otimes c \otimes b) \oplus (a \otimes b) = a \otimes (\overline{1} \oplus c) \otimes b = a \otimes \overline{1} \otimes b = a \otimes b$$

Think of c as the weight of a loop in a path with weight $a \otimes b$.

Shortest paths example, $(\mathbb{N}^{\infty},\ \min,\ +)$



The adjacency matrix

$$\mathbf{A} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{array} \right] \end{array}$$

Note that the longest shortest path is $(1,\ 0,\ 2,\ 3)$ of length 3 and weight 7.

$(\min, +)$ example

Our theorem tells us that $\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{A}^{(4)}$

$$\mathbf{A}^* = \mathbf{A}^{(4)} = \mathbf{I} \ \min\ \mathbf{A}\ \min\ \mathbf{A}^2\ \min\ \mathbf{A}^3\ \min\ \mathbf{A}^4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

$(\min, +)$ example

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \infty & \underline{2} & \underline{1} & 6 & \infty \\ \underline{2} & \infty & 5 & \infty & \underline{4} \\ \underline{1} & 5 & \infty & \underline{4} & \underline{3} \\ 6 & \infty & \underline{4} & \infty & \infty \\ \infty & \underline{4} & \underline{3} & \infty & \infty \end{bmatrix} \end{array}$$

$$\mathbf{A}^3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 8 & 4 & 3 & 8 & 10 \\ 4 & 8 & 7 & \underline{7} & 6 \\ 3 & 7 & 8 & 6 & 5 \\ 8 & \underline{7} & 6 & 11 & 10 \\ 10 & 6 & 5 & 10 & 12 \end{bmatrix} \end{array}$$

$$\mathbf{A}^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2 & 6 & 7 & \underline{5} & \underline{4} \\ 6 & 4 & \underline{3} & 8 & 8 \\ 7 & \underline{3} & 2 & 7 & 9 \\ \underline{5} & 8 & 7 & 8 & \underline{7} \\ \underline{4} & 8 & 9 & \underline{7} & 6 \end{bmatrix} \end{array}$$

$$\mathbf{A}^4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 4 & 8 & 9 & 7 & 6 \\ 8 & 6 & 5 & 10 & 10 \\ 9 & 5 & 4 & 9 & 11 \\ 7 & 10 & 9 & 10 & 9 \\ 6 & 10 & 11 & 9 & 8 \end{bmatrix} \end{array}$$

First appearance of final value is in red and underlined. Remember: we are looking at all paths of a given length, even those with cycles!

A "better" way — our basic algorithm

$$\begin{array}{rcl} \mathbf{A}^{\langle 0 \rangle} & = & \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} & = & \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{array}$$

Lemma

$$\mathbf{A}^{\langle k \rangle} = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

back to (min, +) example

$$\mathbf{A}^{\langle 1 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 6 & \infty \\ 2 & 0 & 5 & \infty & 4 \\ 1 & 5 & 0 & 4 & 3 \\ 6 & \infty & 4 & 0 & \infty \\ \infty & 4 & 3 & \infty & 0 \end{array}\right] \end{array} \quad \mathbf{A}^{\langle 3 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 8 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 8 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

A note on $\mathbf{A}$ vs. $\mathbf{A} \oplus \mathbf{I}$

Lemma

If $\oplus$ is idempotent, then

$$(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}.$$

Proof. Base case: When $k = 0$ both expressions are $\mathbf{I}$.
Assume $(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}$. Then

$$\begin{aligned} (\mathbf{A} \oplus \mathbf{I})^{k+1} &= (\mathbf{A} \oplus \mathbf{I})(\mathbf{A} \oplus \mathbf{I})^k \\ &= (\mathbf{A} \oplus \mathbf{I})\mathbf{A}^{(k)} \\ &= \mathbf{A}\mathbf{A}^{(k)} \oplus \mathbf{A}^{(k)} \\ &= \mathbf{A}(\mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^k) \oplus \mathbf{A}^{(k)} \\ &= \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\ &= \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\ &= \mathbf{A}^{(k+1)} \end{aligned}$$