L11: Algebraic Path Problems with applications to Internet Routing
Lecture 7

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Solving (some) equations

Theorem 6.1

If $\mathbf{A}$ is q-stable, then $\mathbf{A}^*$ solves the equations

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{I}$$

and

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{I}.$$

For example, to show $\mathbf{L} = \mathbf{A}^*$ solves the first equation:

$$\begin{aligned}
\mathbf{A}^* &= \mathbf{A}^{(q)} \\
&= \mathbf{A}^{(q+1)} \\
&= \mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A} \oplus \mathbf{I} \\
&= \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I}) \oplus \mathbf{I} \\
&= \mathbf{A}\mathbf{A}^{(q)} \oplus \mathbf{I} \\
&= \mathbf{A}\mathbf{A}^* \oplus \mathbf{I}
\end{aligned}$$

Note that if we replace the assumption "$\mathbf{A}$ is q-stable" with "$\mathbf{A}^*$ exists," then we require that $\otimes$ distributes over infinite sums.

A more general result

Theorem Left-Right

If $\mathbf{A}$ is q-stable, then $\mathbf{L} = \mathbf{A}^*\mathbf{B}$ solves the equation

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{B}$$

and $\mathbf{R} = \mathbf{B}\mathbf{A}^*$ solves

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{B}.$$

For the first equation:

$$\begin{aligned}
\mathbf{A}^*\mathbf{B} &= \mathbf{A}^{(q)}\mathbf{B} \\
&= \mathbf{A}^{(q+1)}\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^q \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{(q)}\mathbf{B}) \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^*\mathbf{B}) \oplus \mathbf{B}
\end{aligned}$$

The "best" solution

Suppose $\mathbf{Y}$ is a matrix such that

$$\mathbf{Y} = \mathbf{A}\mathbf{Y} \oplus \mathbf{I}$$

$$\begin{aligned}
\mathbf{Y} &= \mathbf{A}\mathbf{Y} \oplus \mathbf{I} \\
&= \mathbf{A}^1\mathbf{Y} \oplus \mathbf{A}^{(0)} \\
&= \mathbf{A}((\mathbf{A}\mathbf{Y} \oplus \mathbf{I})) \oplus \mathbf{I} \\
&= \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A} \oplus \mathbf{I} \\
&= \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A}^{(1)} \\
\vdots \quad & \quad \vdots \quad \vdots \\
&= \mathbf{A}^{k+1}\mathbf{Y} \oplus \mathbf{A}^{(k)}
\end{aligned}$$

If $\mathbf{A}$ is q-stable and $q < k$, then

$$\mathbf{Y} = \mathbf{A}^k\mathbf{Y} \oplus \mathbf{A}^*$$

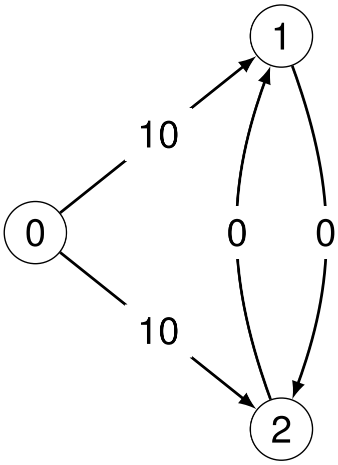
$$\mathbf{Y} \trianglelefteq^{L}_{\oplus} \mathbf{A}^*$$

and if $\oplus$ is idempotent, then

$$\mathbf{Y} \leqslant^{L}_{\oplus} \mathbf{A}^*$$

So $\mathbf{A}^*$ is the largest solution. What does this mean in terms of the sp semiring?

Example with zero weighted cycles using sp semiring



$$\mathbf{A} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \infty & 10 & 10 \\ 1 & \infty & \infty & 0 \\ 2 & \infty & 0 & \infty \end{array}$$

$\mathbf{A}^*$ ($= \mathbf{A} \oplus \mathbf{I}$ in this case) solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

But so does this (dishonest) matrix!

$$\mathbf{F} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 9 & 9 \\ 1 & \infty & 0 & 0 \\ 2 & \infty & 0 & 0 \end{array}$$

For example :

$$\begin{array}{ll} & (\mathbf{FA} \oplus \mathbf{I})(0, 1) \\ = & \min_{q \in \{0,1,2\}} \mathbf{F}(0, q) + \mathbf{A}(q, 1) \\ = & \min(0 + 10, 9 + \infty, 9 + 0) \\ = & 9 \\ = & \mathbf{F}(0, 1) \end{array}$$

Recall our basic iterative algorithm

$$\begin{array}{lcl} \mathbf{A}^{\langle 0 \rangle} & = & \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} & = & \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{array}$$

A closer look ...

$$\begin{array}{lcl} \mathbf{A}^{\langle k+1 \rangle}(i, j) & = & \mathbf{I}(i, j) \oplus \bigoplus_{u} \mathbf{A}(i, u)\mathbf{A}^{\langle k \rangle}(u, j) \\ & = & \mathbf{I}(i, j) \oplus \bigoplus_{(i,u) \in E} \mathbf{A}(i, u)\mathbf{A}^{\langle k \rangle}(u, j) \end{array}$$

This is the basis of distributed Bellman-Ford algorithms (as in RIP and BGP) — a node i computes routes to a destination j by applying its link weights to the routes learned from its immediate neighbors. It then makes these routes available to its neighbors and the process continues...

What if we start iteration in an arbitrary state **M**?

In a distributed environment the topology (captured here by **A**) can change and the state of the computation can start in an arbitrary state (with respect to a new **A**).

$$
\begin{aligned}
\mathbf{A}_{\mathbf{M}}^{\langle 0 \rangle} &= \mathbf{M} \\
\mathbf{A}_{\mathbf{M}}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} \oplus \mathbf{I}
\end{aligned}
$$

Theorem

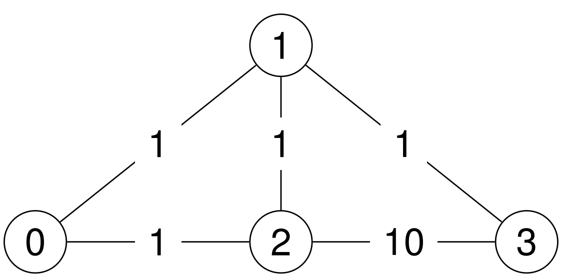
For $1 \leqslant k$,

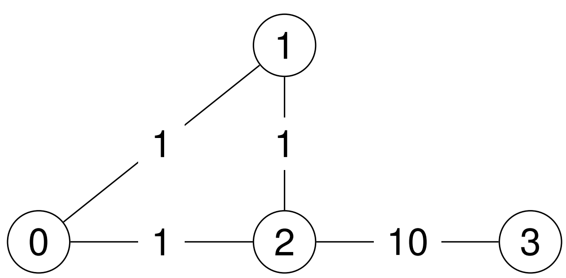
$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{(k-1)}$$

If **A** is q-stable and $q < k$, then

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{*}$$

RIP-like example — counting to convergence (1)





Adjacency matrix $\mathbf{A}_1$

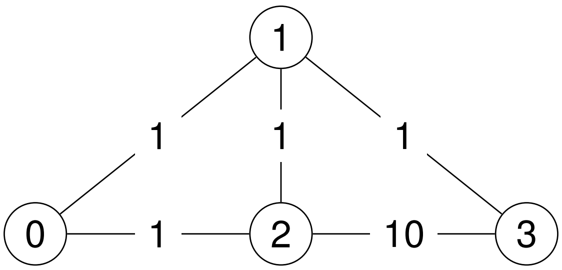
$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
\infty & 1 & 1 & \infty \\
1 & \infty & 1 & 1 \\
1 & 1 & \infty & 10 \\
\infty & 1 & 10 & \infty
\end{array}\right]
\end{array}
$$

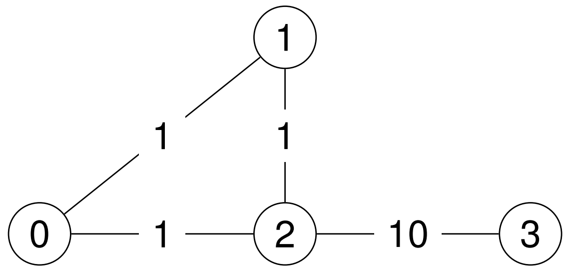
Adjacency matrix $\mathbf{A}_2$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
\infty & 1 & 1 & \infty \\
1 & \infty & 1 & \infty \\
1 & 1 & \infty & 10 \\
\infty & \infty & 10 & \infty
\end{array}\right]
\end{array}
$$

See RFC 1058.

RIP-like example — counting to convergence (2)





The solution $\mathbf{A}_1^*$

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 2 \\
2 & 1 & 2 & 0
\end{array}\right]
\end{array}
$$

The solution $\mathbf{A}_2^*$

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
0 & 1 & 1 & 11 \\
1 & 0 & 1 & 11 \\
1 & 1 & 0 & 10 \\
11 & 11 & 10 & 0
\end{array}\right]
\end{array}
$$

RIP-like example — counting to convergence (3)

The scenario: we arrived at $\mathbf{A}_1^*$, but then links $\{(1,3),\ (3,1)\}$ fail. So we start iterating using the new matrix $\mathbf{A}_2$.

Let $\mathbf{B}_K$ represent $\mathbf{A}_2{}_{\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to convergence (4)

$$\mathbf{B}_0 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_3 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_4 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_5 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

RIP-like example — counting to convergence (5)

$$\mathbf{B}_6 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 7 \\ 1 & 0 & 1 & 7 \\ 1 & 1 & 0 & 7 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

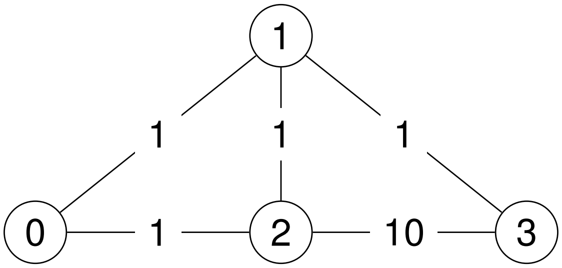
$$\mathbf{B}_7 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 8 \\ 1 & 0 & 1 & 8 \\ 1 & 1 & 0 & 8 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

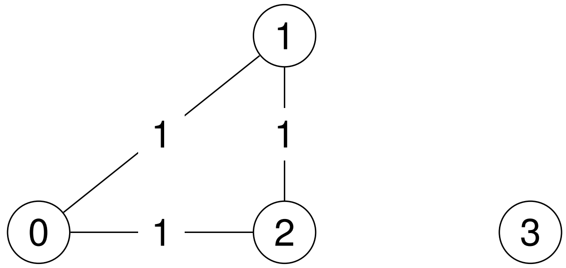
$$\mathbf{B}_8 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 9 \\ 1 & 0 & 1 & 9 \\ 1 & 1 & 0 & 9 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_9 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 10 \\ 1 & 0 & 1 & 10 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_{10} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

RIP-like example — counting to infinity (1)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array}$$

The solution $\mathbf{A}_3^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & \infty \\ 1 & 0 & 1 & \infty \\ 1 & 1 & 0 & \infty \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

Now let $\mathbf{B}_K$ represent $\mathbf{A}_{3\,\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to infinity (2)

$$\mathbf{B}_0 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{376} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 377 \\ 1 & 0 & 1 & 377 \\ 1 & 1 & 0 & 377 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 999 \\ 1 & 0 & 1 & 999 \\ 1 & 1 & 0 & 999 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

RIP-like example — What's going on?

Recall

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle}(i,\ j) = \mathbf{A}^k\mathbf{M}(i,\ j) \oplus \mathbf{A}^*(i,\ j)$$

- $\mathbf{A}^*$(i, j) may be arrived at very quickly
- but $\mathbf{A}^k\mathbf{M}(i,\ j)$ may be better until a very large value of k is reached (counting to convergence)
- or it may always be better (counting to infinity).

Solutions?

- RIP: $\infty = 16$
- In the next lecture we will explore various ways of adding paths to metrics and eliminating those paths with loops ....