L11: Algebraic Path Problems with applications to Internet Routing
Lecture 15

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Path Weight with functions on arcs?

For graph $G = (V,\ E)$, and arc path $p = (u_0,\ u_1)(u_1,\ u_2)\cdots(u_{k-1},\ u_k)$.

Functions on arcs: two natural ways to do this...

Weight function $w : E \rightarrow (S \rightarrow S)$. Let $f_j = w(u_{j-1}, u_j)$.

$$\begin{array}{rcl} w_a^L(p) & = & f_1(f_2(\cdots f_k(a)\cdots)) \quad = \quad (f_1 \circ f_2 \circ \cdots \circ f_k)(a) \\ w_a^R(p) & = & f_k(f_{k-1}(\cdots f_1(a)\cdots)) \quad = \quad (f_k \circ f_{k-1} \circ \cdots \circ f_1)(a) \end{array}$$

How can we "make this work" for path problems?

Algebra of Monoid Endomorphisms (AME) (See Gondran and Minoux 2008)

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \to S, \overline{0})$ is an algebra of monoid endomorphisms (AME) if

- $\forall f \in F, \; f(\overline{0}) = \overline{0}$
- $\forall f \in F, \; \forall b, c \in S, \; f(b \oplus c) = f(b) \oplus f(c)$

I will declare these as optional

- $\forall f, g \in F, \; f \circ g \in F$ (closed)
- $\exists i \in F, \; \forall s \in S, \; i(s) = s$
- $\exists \omega \in F, \; \forall n \in N, \; \omega(n) = \overline{0}$

Note: as with semirings, we may have to drop some of these axioms in order to model Internet routing ...

So why do we want AMEs?

Each (closed with ω and i) AME can be viewed as a semiring of functions. Suppose $(S, \oplus, F, \overline{0})$ is an algebra of monoid endomorphisms. We can turn it into a semiring

$$\mathbb{F} = (F, \hat{\oplus}, \circ, \omega, i)$$

where $(f \; \hat{\oplus} \; g)(a) = f(a) \oplus g(a)$ and $(f \; \circ \; g)(a) = f(g(a))$.

But functions are hard to work with....

- All algorithms need to check equality over elements of a semiring
- $f = g$ means $\forall a \in S, \; f(a) = g(a)$
- S can be very large, or infinite ....

How do we represent a set of functions $F \subseteq S \rightarrow S$?

Assume we a set L and a function

$$\rhd \in L \rightarrow (S \rightarrow S).$$

We normally write $l \rhd s$ rather than $\rhd(l)(s)$. We think of $l \in L$ as the index for a function $f_l(s) = l \rhd s$. In this way $(L, \rhd)$ can be used to represent the set of functions

$$F = \{f_l = \lambda s.(l \rhd s) \mid l \in L\}.$$

Indexed Algebra of Monoid Endomorphisms (IAME)

Let $(S, \oplus, \overline{0})$ be a commutative and idempotent monoid.

A (left) IAME $(S, L, \oplus, \rhd, \overline{0})$

- $\rhd \in L \rightarrow (S \rightarrow S)$
- $\forall l \in L, \ l \rhd \overline{0} = \overline{0}$
- $\exists l \in L, \ \forall s \in S, \ l \rhd s = s$
- $\exists l \in L, \ \forall s \in S, \ l \rhd s = \overline{0}$
- $\forall l \in L, \ \forall n, m \in S, \ l \rhd (n \oplus m) = (l \rhd n) \oplus (l \rhd m)$

When we need closure? Not very often! If needed, it would be

$$\forall l_1, l_2 \in L, \exists l_3 \in L, \forall s \in S, l_3 \rhd s = l_1 \rhd (l_2 \rhd s)$$

IAME of Matrices

Given a left IAME $(S,\ L,\ \oplus,\ \rhd,\ \overline{0})$ define the left IAME of matrices

$$(\mathbb{M}_n(S),\ \mathbb{M}_n(L),\ \oplus,\ \rhd,\ \mathbf{J}).$$

For all i, j we have $\mathbf{J}(i, j) = \overline{0}$. For $\mathbf{A} \in \mathbb{M}_n(L)$ and $\mathbf{B}, \mathbf{C} \in \mathbb{M}_n(S)$ define

$$\begin{aligned}(\mathbf{B} \oplus \mathbf{C})(i,\ j) &= \mathbf{B}(i,\ j) \oplus \mathbf{C}(i,\ j) \\ (\mathbf{A} \rhd \mathbf{B})(i,\ j) &= \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i,\ q) \rhd \mathbf{B}(q,\ j)\end{aligned}$$

Solving (some) equations. Left version here ...

We will be interested in solving for **L** equations of the form

$$\mathbf{L} = (\mathbf{A} \rhd \mathbf{L}) \oplus \mathbf{B}$$

Let

$$\begin{aligned}\mathbf{A} \rhd^0 \mathbf{B} &= \mathbf{B} \\ \mathbf{A} \rhd^{k+1} \mathbf{B} &= \mathbf{A} \rhd (\mathbf{A} \rhd^k \mathbf{B})\end{aligned}$$

and

$$\mathbf{A} \rhd^{(k)} \mathbf{B} = \mathbf{A} \rhd^0 \mathbf{B} \oplus \mathbf{A} \rhd^1 \mathbf{B} \oplus \mathbf{A} \rhd^2 \mathbf{B} \oplus \cdots \oplus \mathbf{A} \rhd^k \mathbf{B}$$

$$\mathbf{A} \rhd^* \mathbf{B} = \mathbf{A} \rhd^0 \mathbf{B} \oplus \mathbf{A} \rhd^1 \mathbf{B} \oplus \mathbf{A} \rhd^2 \mathbf{B} \oplus \cdots \oplus \mathbf{A} \rhd^k \mathbf{B} \oplus \cdots$$

Key result (again)

q stability

If there exists a q such that for all $\mathbf{B}$, $\mathbf{A} \rhd^{(q)} \mathbf{B} = \mathbf{A} \rhd^{(q+1)} \mathbf{B}$, then $\mathbf{A}$ is q-stable. Therefore, $\mathbf{A} \rhd^{*} \mathbf{B} = \mathbf{A} \rhd^{(q)} \mathbf{B}$.

Theorm

If $\mathbf{A}$ is q-stable, then $\mathbf{L} = \mathbf{A} \rhd^{*} (\mathbf{B})$ solves the equation

$$\mathbf{L} = (\mathbf{A} \rhd \mathbf{L}) \ \oplus \ \mathbf{B}.$$

Something familiar : Lexicographic product

$$(S,\ L_S,\ \oplus_S,\ \rhd_S) \vec{\times} (T,\ L_T,\ \oplus_T,\ \rhd_T) \equiv (S \times T,\ \ L_S \times L_T,\ \oplus_S \vec{\times} \oplus_T,\ \rhd_S \times \rhd_T)$$

Theorem

$$\begin{gathered}\mathbb{D}((S,\ L_S,\ \oplus_S,\ \rhd_S) \ \vec{\times}\ (T,\ L_T,\ \oplus_T,\ \rhd_T)) \\ \iff \\ \mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T) \\ \wedge\ (\mathbb{C}(S,\ L_S,\ \rhd_S) \vee \mathbb{K}(T,\ L_T,\ \rhd_T))\end{gathered}$$

Where

$$\begin{array}{rcl}\mathbb{D}(S,\ L,\ \oplus,\ \rhd) & \equiv & \forall a, b \in S,\ l \in L,\ l \rhd (a \oplus b) = (l \rhd a) \oplus (l \rhd b) \\ \mathbb{C}(S,\ L,\ \rhd) & \equiv & \forall a, b \in S,\ l \in L,\ l \rhd a = l \rhd b \implies a = b \\ \mathbb{K}(S,\ L,\ \rhd) & \equiv & \forall a, b \in S,\ l \in L,\ l \rhd a = l \rhd b\end{array}$$

Something new: Functional Union

$$(S,\ L_1,\ \oplus,\ \rhd_1) +_{\mathrm{m}} (S,\ L_2,\ \oplus,\ \rhd_2) = (S,\ L_1 \uplus L_2,\ \oplus,\ \rhd_1 \uplus \rhd_2)$$

Where

$$\begin{array}{rcl} \mathrm{inl}(l)\ (\rhd_1 \uplus \rhd_2)\ s & = & l\ \rhd_1\ s \\ \mathrm{inr}(l)\ (\rhd_1 \uplus \rhd_2)\ s & = & l\ \rhd_2\ s \end{array}$$

Fact

$$\mathbb{D}((S,\ L_1,\ \oplus,\ \rhd_1) +_{\mathrm{m}} (S,\ L_2,\ \oplus,\ \rhd_2))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S,\ L_1,\ \oplus,\ \rhd_1) \wedge \mathbb{D}(S,\ L_2,\ \oplus,\ \rhd_2)$$

Left and Right

$\mathrm{right}(S,\ \oplus) \equiv (S,\ \{R\},\ \oplus, \textbf{right})$

$$R\ \textbf{right}\ s = s$$

$\mathrm{left}(S,\ \oplus) \equiv (S,\ S,\ \oplus,\ \textbf{left})$

$$s_1\ \textbf{left}\ s_2 = s_1$$

The following are always hold.

$$\mathbb{D}(\mathrm{right}(S,\ \oplus))$$
$$\mathbb{IP}(S,\ \oplus) \Rightarrow \mathbb{D}(\mathrm{left}(S,\ \oplus))$$
$$\mathbb{C}(\mathrm{right}(S,\ \oplus))$$
$$\mathbb{K}(\mathrm{left}(S,\ \oplus))$$

Scoped Product (Think iBGP/eBGP)

$$(S,\ L_S,\ \oplus_S,\ \rhd_S)\ \Theta\ (T,\ L_T,\ \oplus_T,\ \rhd_T)$$
$$\equiv$$
$$((S,\ L_S,\ \oplus_S,\ \rhd_S)\ \vec{\times}\ \mathrm{left}(T,\ \oplus_T))\ +_{\mathrm{m}}\ (\mathrm{right}(S,\ \oplus_S)\ \vec{\times}\ (T,\ L_T,\ \oplus_T,\ \rhd_T))$$

Between regions ($\lambda \in L_S$, $s \in S$, $t_1, t_2 \in T$)

$$\mathrm{inl}(\lambda,\ t_2) \rhd (s,\ t_1) = (\lambda \rhd_S s,\ t_2)$$

Within regions ($\lambda \in L_T$, $s \in S$, $t \in T$)

$$\mathrm{inr}(R,\ \lambda) \rhd (s,\ t) = (s,\ \lambda \rhd_T t)$$

Theorem. If $\mathbb{IP}(T,\ \oplus_T)$, then

$$(\mathbb{D}((S,\ L_S,\ \oplus_S,\ \rhd_S)\ \Theta\ (T,\ L_T,\ \oplus_T,\ \rhd_T))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T))$$

$$\begin{aligned}
& \mathbb{D}(((S,\ L_S,\ \oplus_S,\ \rhd_S)\ \vec{\times}\ \mathrm{left}(T,\ \oplus_T)) \\
& \qquad +_{\mathrm{m}}\ (\mathrm{right}(S,\ \oplus_S)\ \vec{\times}\ (T,\ L_T,\ \oplus_T,\ \rhd_T))) \\
\Longleftrightarrow\ & \mathbb{D}((S,\ L_S,\ \oplus_S,\ \rhd_S)\ \vec{\times}\ \mathrm{left}(T,\ \oplus_T)) \\
& \qquad \wedge\ \mathbb{D}((\mathrm{right}(S,\ \oplus_S))\ \vec{\times}\ (T,\ L_T,\ \oplus_T,\ \rhd_T)) \\
\Longleftrightarrow\ & \mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(\mathrm{left}(T,\ \oplus_T)) \\
& \qquad \wedge\ (\mathbb{C}(S,\ L_S,\ \rhd_S) \vee \mathbb{K}(\mathrm{left}(T,\ \oplus_T))) \\
& \qquad \wedge\ \mathbb{D}(\mathrm{right}(S,\ \oplus_S)) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T) \\
& \qquad \wedge\ (\mathbb{C}(\mathrm{right}(S,\ \oplus_S)) \vee \mathbb{K}(T,\ L_T,\ \rhd_T)) \\
\Longleftrightarrow\ & \mathbb{D}(S,\ L_S,\ \oplus_S,\ \rhd_S) \wedge \mathbb{D}(T,\ L_T,\ \oplus_T,\ \rhd_T)
\end{aligned}$$

Homework 3 : Shortest Paths with Gains and Losses

$$\text{spgl} \equiv (\mathbb{R} \times \mathbb{R},\ \oplus,\ \bullet)$$

Where

$$(s_1,\ t_1) \oplus (s_2,\ t_2) \quad \equiv \quad \begin{cases} (s_1,\ t_1 \min t_2) & (\text{if } s_1 = s_1) \\ (s_1,\ t_1) & (\text{if } s_1 = s_1 \min s_2 \neq s_2) \\ (s_2,\ t_2) & (\text{if } s_1 \neq s_1 \min s_2 = s_2) \end{cases}$$

$$(c_1,\ n_1) \bullet (c_2,\ n_2) \quad \equiv \quad (c_1 + n_1 c_2,\ n_1 n_2)$$

Homework 3 : Calculating path weights in spgl

$$\begin{array}{cl} & ((c_1,\ n_1) \bullet (c_2,\ n_2)) \bullet (c_3,\ n_3) \\ = & (c_1 + n_1 c_2,\ n_1 n_2) \bullet (c_3,\ n_3) \\ = & (c_1 + n_1 c_2 + n_1 n_2 c_3,\ n_1 n_2 n_3) \end{array}$$

$$\begin{array}{cl} & (c_1,\ n_1) \bullet (c_2,\ n_2) \bullet (c_3,\ n_3) \bullet \cdots \bullet (c_k,\ n_k) \\ = & (c_1 + n_1 c_2 + n_1 n_2 c_3 + \cdots + n_1 n_2 n_3 \cdots n_{k-1} c_k,\ n_1 n_2 n_3 \cdots n_k) \end{array}$$

Homework 3 : Shortest Paths with Discounting

Assume $0 < \delta < 1$.

$$\mathrm{spd} \equiv (\mathbb{R} \times \mathbb{N},\ \oplus,\ \diamond)$$

Where

$$(s_1,\ t_1) \oplus (s_2,\ t_2) \ \equiv\ \begin{cases} (s_1,\ t_1 \min t_2) & (\text{if } s_1 = s_1) \\ (s_1,\ t_1) & (\text{if } s_1 = s_1 \min s_2 \neq s_2) \\ (s_2,\ t_2) & (\text{if } s_1 \neq s_1 \min s_2 = s_2) \end{cases}$$

$$(c_1,\ n_1) \diamond (c_2,\ n_2) \ \equiv\ (c_1 + c_2\delta^{n_1},\ n_1 + n_2)$$

Homework 3 : Calculating path weights in spd

$$\begin{array}{cl} & ((c_1,\ n_1)\ \diamond (c_2,\ n_2)) \diamond\ (c_3,\ n_3) \\ = & (c_1 + c_2\delta^{n_1},\ n_1 + n_2) \diamond\ (c_3,\ n_3) \\ = & (c_1 + c_2\delta^{n_1} + c_3\delta^{(n_1+n_2)},\ n_1 + n_2 + n_3) \\ = & (c_1 + c_2\delta^{n_1} + c_3\delta^{n_1}\delta^{n_2},\ n_1 + n_2 + n_3) \end{array}$$

Homework 3 : Can we generalise this kind of construction?

Given semigroups $(S, \otimes_S)$ and $(T, \otimes_T)$, and a function

$$\rhd \in T \to (S \to S)$$

Write $t \rhd s$ instead of $\rhd(t)(s)$.

Define the semi-direct product $\rtimes_{\rhd}$ on $S \times T$ as

$$(s_1,\ t_1)\ \rtimes_{\rhd}\ (s_2,\ t_2)\ \equiv\ (s_1 \otimes_S (t_1 \rhd s_2),\ t_1 \otimes_T t_2)$$

Homework 3 : A new construction

Assume $\oplus$ is commutative and selective.

$$(S,\ \oplus_S,\ \otimes_S) \rtimes_{\rhd} (T,\ \oplus_T,\ \otimes_T) \equiv (S \times T,\ \oplus,\ \rtimes_{\rhd})$$

where $\oplus = \oplus_T \vec{\times} \oplus_T$ (lexicographic product).

Homework 3 : Does this work?

spgl

Let

$$n \rhd c \equiv n \times c$$

then

$$\mathrm{spgl} = (\mathbb{R},\ \min,\ +) \rtimes_{\rhd} (\mathbb{R},\ \min,\ \times)$$

spd

Assume $0 < \delta < 1$ and

$$n \rhd c \equiv c \times \delta^n,$$

then

$$\mathrm{spd} = (\mathbb{R},\ \min,\ +) \rtimes_{\rhd} (\mathbb{N},\ \min,\ +)$$

Homework 3 : Problems

Question 1 (30 marks)

When is $(S \times T, \rtimes_{\rhd})$ a semigroup?

Question 2 (35 marks)

When is $(S, \oplus_S, \otimes_S) \rtimes_{\rhd} (T, \oplus_T, \otimes_T)$ left distributive?

Question 3 (35 marks)

When is $(S, \oplus_S, \otimes_S) \rtimes_{\rhd} (T, \oplus_T, \otimes_T)$ right distributive?