

# This lecture

Some category theory relevant  
to modelling type theories with

dependent types

Will restrict attention to  
what it looks like just in **Set**  
rather than in full generality

[See e.g. : M. Hofmann, "Syntax & Semantics of Dependent  
Types", pp 79-130 of A. Pitts & P. Dybjer, "Semantics &  
Logics of Computation (CUP, 1997) ]

Simple types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T$$

# Simple types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T$$

# Dependent types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T(x_1, \dots, x_n)$$

# & more generally

$$x_1 : T_1, x_2 : T_2(x_1), x_3 : T_3(x_1, x_2), \dots \vdash t : T(x_1, x_2, \dots)$$

If types denote sets,  
then a dependent type

$$T(x) \quad [x : T']$$

should denote an

indexed family of sets

$$E = (E_i \mid i \in I)$$

i.e.  $E$  is a function  $I \rightarrow \text{obj}(\text{Set})$

For each  $I \in \text{obj Set}$ , let  $\boxed{\text{Set}^I}$  be the category with

- $\text{obj}(\text{Set}^I) \triangleq (\text{obj Set})^I$   
so objects are  $I$ -indexed families of sets  
 $X = (X_i \mid i \in I)$

- morphisms  $f: X \rightarrow Y$  in  $\text{Set}^I$  are  $I$ -indexed families of functions  
 $f = (f_i \in \text{Set}(X_i, Y_i) \mid i \in I)$

Composition in Set

- composition:  $(g \circ f) = (g_i \circ f_i \mid i \in I)$   
identities:  $\text{id}_X = (\text{id}_{X_i} \mid i \in I)$

Identity in Set (4.5)

For each  $p: I \rightarrow J$  in  $\text{Set}$ , let

$$p^*: \text{Set}^J \rightarrow \text{Set}^I$$

be the functor

$$p^* \left( \begin{array}{c} Y_j \\ \downarrow f_j \\ Y'_j \end{array} \middle| j \in J \right) \cong \left( \begin{array}{c} Y_{p(i)} \\ \downarrow f_{p(i)} \\ Y'_{p(i)} \end{array} \middle| i \in I \right)$$

i.e.  $p^*$  takes  $J$ -indexed families of sets/functions to  $I$ -indexed ones by precomposing with  $p$

# Dependent products of families of sets

for  $I, J \in \text{obj Set}$ , projection  $\pi_1: I \times J \rightarrow I$   
gives a functor  $\pi_1^*: \text{Set}^I \rightarrow \text{Set}^{I \times J}$

Theorem  $\pi_1^*$  has a left adjoint  
 $\Sigma: \text{Set}^{I \times J} \rightarrow \text{Set}^I$

Proof For each  $E \in \text{Set}^{I \times J}$  we give  
 $\Sigma E \in \text{Set}^I$  &  $\eta_E: E \rightarrow \pi_1^*(\Sigma E)$   
with required universal property ...

For each  $E \in \text{Set}^{I \times J}$ , we define  $\Sigma E \in \text{Set}^I$

by:

$$(\Sigma E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{ (j, e) \mid j \in J \wedge e \in E_{(i,j)} \}$$

(all  $i \in I$ )

and  $\eta_E \in \text{Set}^{I \times J}(E, \pi_i^*(\Sigma E))$  by

$$(\eta_E)_{(i,j)} : E_{(i,j)} \rightarrow (\Sigma E)_i$$
$$e \mapsto (j, e)$$

(all  $(i,j) \in I \times J$ )

Universal property of  $\eta_E : E \rightarrow \pi_1^*(\Sigma E)$  :

Given any  $X \in \text{Set}^I$  &  $f : E \rightarrow \pi_1^*(X)$   
in  $\text{Set}^{I \times J}$ , we have :

existence

$$\begin{array}{ccc}
 E & \xrightarrow{\eta_E} & \pi_1^*(\Sigma E) \\
 & \searrow f & \downarrow \pi_1^*(\hat{f}) \\
 & & \pi_1^*(X)
 \end{array}$$

$$\begin{array}{c}
 \Sigma E \\
 \downarrow \hat{f} \\
 X
 \end{array}$$

← in  $\text{Set}^I$

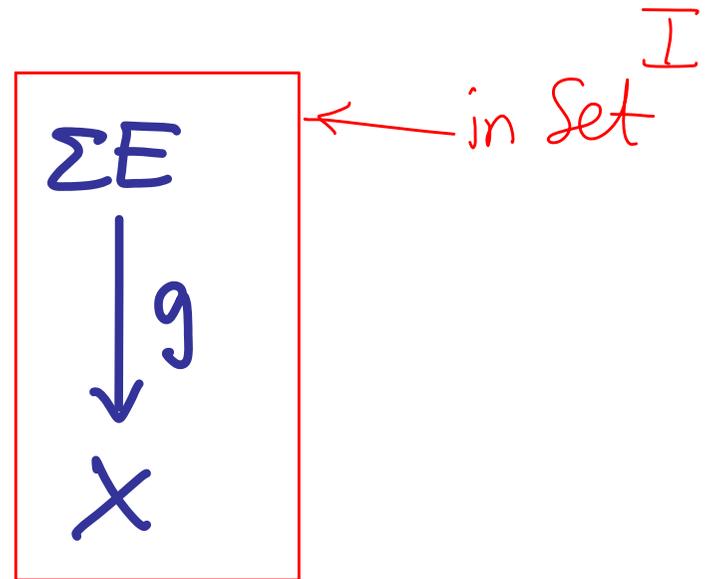
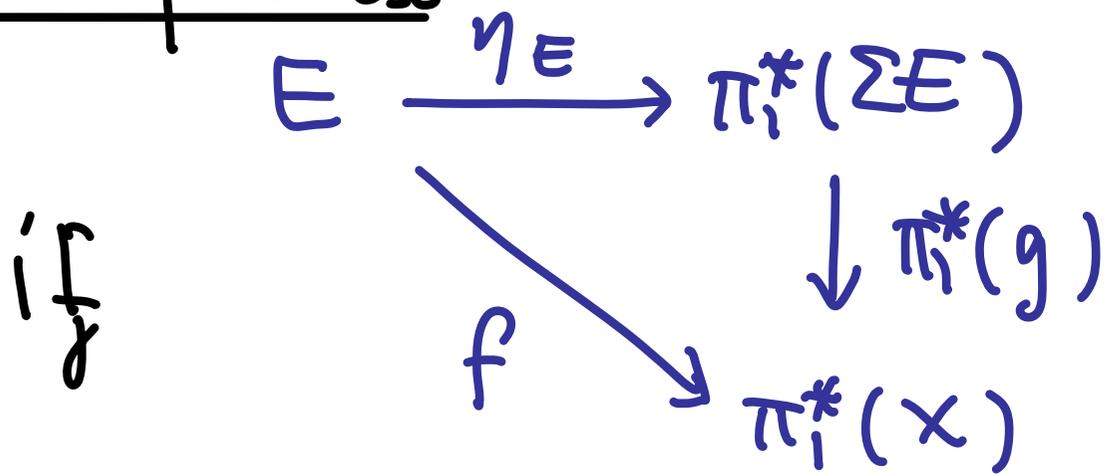
where

$$\hat{f}_i(j, e) \triangleq f_{(ij)}(e) \quad (\text{all } \begin{array}{l} i \in I \\ j \in J \\ e \in E_{(ij)} \end{array})$$

Universal property of  $\eta_E : E \rightarrow \pi_1^*(\Sigma E)$

Given any  $X \in \text{Set}^I$  &  $f : E \rightarrow \pi_1^*(X)$   
 in  $\text{Set}^{I \times J}$ , we have:

uniqueness



then for all  $i \in I$ ,  $(j, e) \in (\Sigma E)_i$ , have

$$\hat{f}_i(j, e) = f_{(ij)}(e) = (\pi_1^* g \circ \eta_E)_{(ij)} e = g_i(j, e) \quad \text{So } g = \hat{f} \quad \square$$

# Dependent functions for families of sets

for  $I, J \in \text{obj Set}$ , projection  $\pi_1: I \times J \rightarrow I$   
gives a functor  $\pi_1^*: \text{Set}^I \rightarrow \text{Set}^{I \times J}$

Theorem  $\pi_1^*$  has a right adjoint  
 $\Pi: \text{Set}^{I \times J} \rightarrow \text{Set}^I$

Proof For each  $E \in \text{Set}^{I \times J}$  we give

$$\Pi E \in \text{Set}^I \quad \& \quad \varepsilon_E: \pi_1^*(\Pi E) \rightarrow E$$

with required universal property ...

For each  $E \in \text{Set}^{I \times J}$ , we define  $\prod E \in \text{Set}^I$

by:

$$(\prod E)_i \triangleq \prod_{j \in J} E_{(i,j)}$$

$$= \{ f \subseteq (\sum E)_i \mid \begin{array}{l} f \text{ is single-valued} \\ \& \text{total} \end{array} \}$$

where  $f \subseteq (\sum E)_i$  is

**single-valued** if  $(\forall j \in J)(\forall e, e' \in E_{(i,j)}) (j, e) \in f \wedge (j, e') \in f \Rightarrow e = e'$

**total** if  $(\forall j \in J)(\exists e \in E_{(i,j)}) (j, e) \in f$

ie. each  $f \in (\prod E)_i$  is a dependently typed function mapping elements  $j \in J$  to elements of  $E_{(i,j)}$  depends on argument  $j$

For each  $E \in \text{Set}^{I \times J}$ , we define  $\Pi E \in \text{Set}^I$

by:

$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)}$$

$$= \{ f \subseteq (\Sigma E)_i \mid \begin{array}{l} f \text{ is single-valued} \\ \& \text{total} \end{array} \}$$

and  $\varepsilon_E \in \text{Set}^{I \times J} (\pi_1^*(\Pi E), E)$  by

$$(\varepsilon_E)_{(i,j)} : (\Pi E)_i \rightarrow E_{(i,j)}$$

$$f \longmapsto f(j)$$

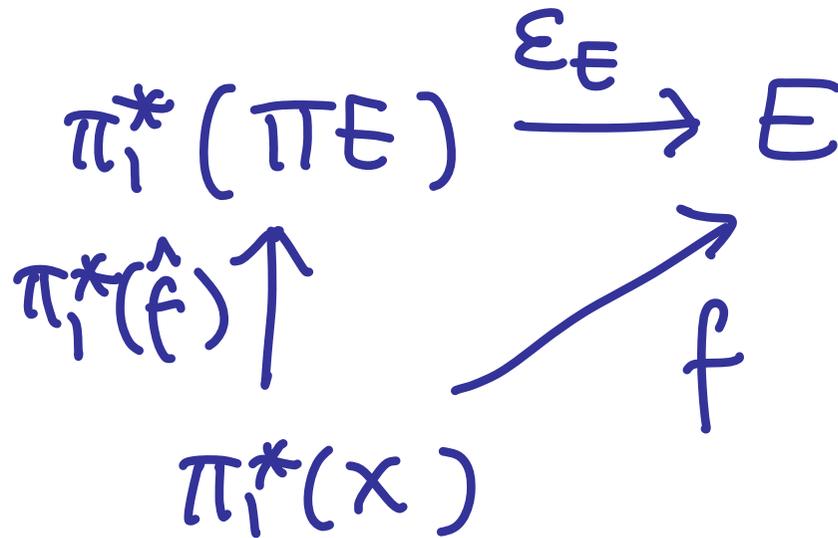
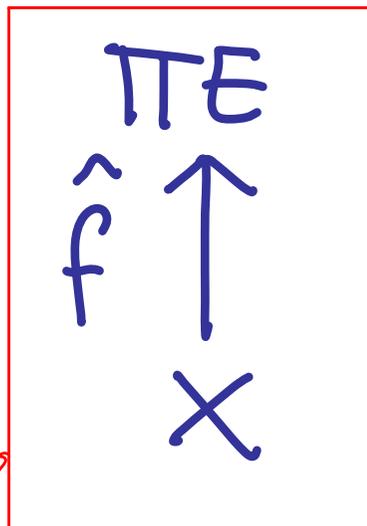
(all  $(i,j) \in I \times J$ )

the unique  
 $e \in E_{(i,j)}$  such  
that  $(j, e) \in f$

# Universal property of $\varepsilon_E : \pi_1^*(\pi E) \rightarrow E$

Given any  $X \in \text{Set}^{\mathbf{I}}$  and  $f : \pi_1^*(X) \rightarrow E$   
in  $\text{Set}^{\mathbf{I} \times \mathbf{J}}$ , we have:

existence



in  $\text{Set}^{\mathbf{I}}$

where  $\hat{f}_i(x) \triangleq \{(j, f_{(i,j)}(x)) \mid j \in \mathbf{J}\}$   
(all  $i \in \mathbf{I}, x \in X_i$ )

# Universal property of $\varepsilon_E : \pi_1^*(\pi E) \rightarrow E$

Given any  $X \in \text{Set}^I$  and  $f : \pi_1^*(X) \rightarrow E$  in  $\text{Set}^{I \times J}$ , we have:

## Uniqueness

if

$$\begin{array}{c} \pi E \\ \uparrow g \\ X \end{array}$$

$$\begin{array}{ccc} \pi_1^*(\pi E) & \xrightarrow{\varepsilon_E} & E \\ \uparrow \pi_1^*(g) & \nearrow f & \\ \pi_1^*(X) & & \end{array}$$

in  $\text{Set}^I$

then  $f_i x_j = f_{(i,j)} x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)} x = g_i x_j$

so  $g = \hat{f}$

