Adjoint functors

Categories, functors & natural transformations were invented (by Eilenberg & MacLane) in order to formalize "adjoint situations".
They appear everywhere in mathematics, logic and (hence) CS.

Examples that we have seen ...

Binary product in $\mathbb{C}$

$$\frac{(Z,Z) \to (X,Y)}{Z \to X \times Y}$$

morphisms in $\mathbb{C} \times \mathbb{C}$

morphisms in $\mathbb{C}$

bijective correspondence:

$$\mathbb{C}\times\mathbb{C}\big((Z,Z),(X,Y)\big) \cong \mathbb{C}(Z, X\times Y)$$

$$(f,g) \longmapsto \langle f,g\rangle$$

$$(\pi_1\circ h, \pi_2\circ h) \longleftarrow\!\shortmid\; h$$

Furthermore, this bijection "is natural in X, Y, Z" (to be explained)

Exponentials in $\mathbb{C}$

$$\frac{Z \times X \longrightarrow Y}{Z \longrightarrow Y^X}$$

morphisms in $\mathbb{C}$

morphisms in $\mathbb{C}$

bijective correspondence

$$\mathbb{C}(Z \times X, Y) \cong \mathbb{C}(Z, Y^X)$$

$$f \longmapsto \mathrm{cur}\, f$$

$$\mathrm{app} \circ (g \times \mathrm{id}_X) \longleftarrow\!\shortmid\; g$$

natural in X, Y, Z

Free monoids

$$\dfrac{\Sigma \to U(M, \cdot, e) \quad \text{in Set}}{F\Sigma \to (M, \cdot, e) \quad \text{in Mon}}$$

free monoid on set Σ
$(\mathrm{List}(\Sigma), \_@\_, \mathrm{nil})$

bijective correspondence

$$\mathrm{Set}(\Sigma, UM) \cong \mathrm{Mon}(F\Sigma, M)$$

$$f \longmapsto \hat{f}$$

$$g \circ i_\Sigma \longleftarrow\!\shortmid\; g$$

natural in Σ & M

Adjunction

<u>Definition</u> An adjunction between two categories $\mathbb{C}$ & $\mathbb{D}$ is specified by

- functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

- bijections

$$\theta_{X,Y} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$$

for each $X \in \operatorname{obj} \mathbb{C}$ & $Y \in \operatorname{obj} \mathbb{D}$
which are natural in X & Y, meaning...

For $\theta_{X,Y} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$

to be "natural in X & Y" means

for all $\begin{cases} u : X' \to X \text{ in } \mathbb{C} \\ v : Y \to Y' \text{ in } \mathbb{D} \end{cases}$

and all $g : F(X) \to Y$ in $\mathbb{D}$

$$X' \xrightarrow{u} X \xrightarrow{\theta_{X,Y}(g)} G(Y) \xrightarrow{Gv} G(Y')$$

$$= \theta_{X',Y'}\left(F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y'\right)$$

For $\theta_{X,Y} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$
to be "natural in X & Y" means
for all $\begin{cases} u : X' \to X \text{ in } \mathbb{C} \\ v : Y \to Y' \text{ in } \mathbb{D} \end{cases}$
and all $g : F(X) \to Y$ in $\mathbb{D}$

$$X' \xrightarrow{u} X \xrightarrow{\theta_{X,Y}(g)} G(Y) \xrightarrow{Gv} G(Y')$$

$$= \theta_{X',Y'} \quad F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y'$$

what has this to do with the concept of natural transformation?

Hom functors

If $\mathbb{C}$ is locally small, then we get a functor

$$\boxed{H_{\mathbb{C}} : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbf{Set}}$$

with $H_{\mathbb{C}}(X,Y) \triangleq \mathbb{C}(X,Y)$ and

$$H_{\mathbb{C}}\Big((X,Y) \xrightarrow{(f,g)} (X',Y') \Big) \triangleq \begin{array}{rcl} \mathbb{C}(X,Y) & \longrightarrow & \mathbb{C}(X',Y') \\ h & \longmapsto & g \circ h \circ f \end{array}$$

$f : X' \to X$ in $\mathbb{C}$

$g : Y \to Y'$ in $\mathbb{C}$

Natural isomorphisms

Given categories and functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathbb{D} \end{tikzcd}$$

a natural isomorphism $\theta : F \cong G$ is simply an isomorphism between F & G in the functor category $\mathbb{D}^{\mathbb{C}}$.

Natural isomorphisms

Given categories and functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathbb{D} \end{tikzcd}$$

<u>FACT</u> If $\theta : F \to G$ is a nat. transf^n and for each $X \in obj\,\mathbb{C}$, $\theta_X : F(X) \to G(X)$ is an isomorphism in $\mathbb{D}$, then $\theta_X^{-1} : G(X) \to F(X)$ gives a nat. transf^n $\theta^{-1} : G \to F$ & $F \cong G$ in $\mathbb{D}^{\mathbb{C}}$.

Given locally small categories $\mathbb{C}$ & $\mathbb{D}$, if we have $\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$ we get functors

$$\begin{tikzcd}
& \mathbb{D}^{op} \times \mathbb{D} \arrow[dr, "H_{\mathbb{D}}"] & \\
\mathbb{C}^{op} \times \mathbb{D} \arrow[ur, "F^{op} \times id"] \arrow[dr, "id \times G"'] & & \mathbf{Set} \\
& \mathbb{C}^{op} \times \mathbb{C} \arrow[ur, "H_{\mathbb{C}}"'] &
\end{tikzcd}$$

An adjunction (F, G, θ) is give by a nat. iso $\theta : H_{\mathbb{D}} \circ (F^{op} \times id) \cong H_{\mathbb{C}} \circ (id \times G)$

Terminology

Given
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd},$$

if there is some $\theta : H_{\mathbb{D}} \circ (F^{op} \times id) \cong H_{\mathbb{C}} \circ (id \times G)$

one says

F is a left adjoint for G

G is a right adjoint for F

and writes $\boxed{F \dashv G}$

Notation associated with an adjunction (F, G, θ)

Given $\begin{cases} g : FX \to Y \\ f : X \to GY \end{cases}$

we write $\begin{cases} \bar{g} \triangleq \theta_{X,Y}(g) : X \to GY \\ \bar{f} \triangleq \theta^{-1}_{X,Y}(f) : FX \to Y \end{cases}$

Thus $\bar{\bar{g}} = g$, $\bar{\bar{f}} = f$ and naturality means

$$\frac{\bar{g} = g,\ \bar{\bar{f}} = f}{v \circ g \circ Fu \;=\; Gv \circ \bar{g} \circ u}$$

The existence of θ is sometimes indicated by writing

$$\boxed{\dfrac{Fx \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY}}$$

The existence of θ is sometimes indicated by writing

$$\theta \;\Big\downarrow\; \boxed{\dfrac{Fx \xrightarrow{g} Y}{x \xrightarrow{\bar{g}} GY}} \;\Big\uparrow\; \theta^{-1}$$

Using this notation, can split the naturality condition for θ into two:

$$\frac{Fx' \xrightarrow{Fu} Fx \xrightarrow{g} Y}{x' \xrightarrow{u} x \xrightarrow{\bar{g}} GY} \qquad \frac{Fx \xrightarrow{g} Y \xrightarrow{v} Y'}{x \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$

12.15