# Hoare Logic and Model Checking

## Model Checking
## Lecture 10: Computation Tree Logic (CTL)

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# Learning outcomes

By the end of this lecture, you should:

- Be familiar with the branching model of time
- Be familiar with CTL syntax and semantics
- Understand CTL semantic equivalence, and why it is important
- Be familiar with important CTL equivalences
- Be familiar with Existential Normal Form

# Branching model of time

# Branching time

CTL's conception of time:

- At each moment in time exactly potentially multiple futures
- Time “branches” into multiple futures at each state
- Quantify over possible futures

CTL therefore describes “state properties” of systems

CTL formulae describe states in transition system

# A note on models

Note: by changing model of time, not changed underlying model

CTL models are based on right-serial transition systems, same as LTL

Changing conception of time:

- Affects properties that can be expressed by formulae
- Affects what CTL formulae describe (states, not paths)

# CTL syntax

# Atomic propositions

Like in LTL, we fix a set $AP$ of **atomic propositions**

We continue to use $p$, $q$, $r$, and so on to range over $AP$

# CTL state and path formulae

Define **state formulae** with the following grammar:

$$
\begin{aligned}
\Phi, \Psi, \Xi &::= \top \mid \bot \mid p \\
&::= \neg\Phi \\
&::= \Phi \wedge \Psi \mid \Phi \vee \Psi \mid \Phi \Rightarrow \Psi \\
&::= \forall\phi \mid \exists\phi
\end{aligned}
$$

and **path formulae** with the following grammar:

$$\phi, \psi, \xi ::= \bigcirc\Phi \mid \Box\Phi \mid \Diamond\Phi \mid \Phi \texttt{ UNTIL } \Psi$$

In semantics of CTL:

- Path formulae are evaluated relative to a path
- State formulae are evaluated relative to a state

# Intuitive explanation of CTL formulae

First line (of state formula grammar):

$$\top \mid \bot \mid p$$

$\top$, $\bot$, and $p$ for $p$ atomic are all primitive CTL state formulae

- $\top$ is the **logical truth** constant (or "true"),
- $\bot$ is the **logical falsity** constant (or "false"),
- $p$ is the embedding of **atomic propositions** into CTL formulae

The last should now be familiar too!

# Intuitive explanation of CTL formulae

Second line (of state formula grammar):

$$\neg\Phi$$

If $\Phi$ is a CTL state formula, then $\neg\Phi$ is a CTL state formula

- $\neg\Phi$ is **negation** of $\phi$ (or "not $\Phi$")

# Intuitive explanation of CTL formulae

Third line (of state formula grammar):

$$\Phi \wedge \Psi \mid \Phi \vee \Psi \mid \Phi \Rightarrow \Psi$$

If $\Phi$ and $\Psi$ are CTL state formulae, then so are $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \Rightarrow \Psi$

- $\Phi \wedge \Psi$ is **conjunction** (or "$\Phi$ and $\Psi$")
- $\Phi \vee \Psi$ is **disjunction** (or "$\Phi$ or $\Psi$")
- $\Phi \Rightarrow \Psi$ is **implication** (or "if $\Phi$ then $\Psi$", or "$\psi$ whenever $\phi$")

# Intuitive explanation of CTL formulae

Last line (of state formula grammar):

$$\forall\phi \mid \exists\phi$$

If $\phi$ and $\psi$ are CTL path formulae, then $\forall\phi$ and $\exists\phi$ are CTL state formulae

- $\forall\phi$ is "$\phi$ along every path that starts here"
- $\exists\phi$ is "$\phi$ along at least one path that starts here", or "there exists a path where $\phi$ holds"

Specific to CTL!

# Intuitive explanation of CTL formulae

Path formula grammar:

$$\bigcirc\Phi \mid \Box\Phi \mid \Diamond\Phi \mid \Phi \ \texttt{UNTIL}\ \Psi$$

If $\Phi$ and $\Psi$ are CTL state formulae, then $\bigcirc\Phi$, $\Box\Phi$, $\Diamond\Phi$, and $\Phi \ \texttt{UNTIL}\ \Psi$ are CTL path formulae

- $\Box\Phi$ is "henceforth $\Phi$", or "from now, always $\Phi$"
- $\Diamond\Phi$ is "at some future point $\Phi$"
- $\bigcirc\Phi$ is "immediately after $\Phi$", or "in the next state $\Phi$"
- $\Phi \ \texttt{UNTIL}\ \Psi$ is "at some future point $\Psi$, but until then $\Phi$"

# Alternative syntax for modalities

Grammar above enforces path formula be "covered" by quantifier

Impossible to construct $\Box\forall\phi$ or $\exists\phi$ $\texttt{UNTIL}$ $\Psi$

Effect is to have

$$\forall\Box\Phi \quad \exists\Box\Phi \quad \forall\Diamond\Phi \quad \exists\Diamond\Phi$$

$$\forall\bigcirc\Phi \quad \exists\bigcirc\Phi \quad \forall(\Phi\ \texttt{UNTIL}\ \Psi) \quad \exists(\Phi\ \texttt{UNTIL}\ \Psi)$$

$\forall\Box$, $\exists\bigcirc$, and so on, are "derived modalities"

# Alternative syntax for modalities

Some collapse grammar of CTL into a single grammar of "formulae"

Less clear (to me, anyway) what is going on:

- $\forall$ and $\exists$ are instructions: "go off and examine paths"
- Path formulae evaluated relative to paths
- State formulae relative to states
- Grammar closer to grammar of CTL$\star$

Might also see (e.g. in "Logic in Computer Science"):

- $A$ and $E$ instead of $\forall$ and $\exists$
- $X$, $G$, $F$, and $U$ instead of $\bigcirc$, $\Box$, $\Diamond$, and `UNTIL`

# Operator precedence

We add parentheses freely to disambiguate

Assign precedence to reduce number of parentheses needed:

- Unary $\neg$, $\forall$, $\exists$, $\Box$, $\Diamond$, and $\bigcirc$ bind most tightly
- After that `UNTIL`
- After that $\vee$ and $\wedge$
- Finally $\Rightarrow$ binds least tightly

# Precedence examples

So:

$$\Phi \Rightarrow \forall \bigcirc \Psi \quad \text{means} \quad \Phi \Rightarrow (\forall(\bigcirc\Psi))$$

$$\Phi \Rightarrow \Psi \vee \exists\Box\Psi \quad \text{means} \quad \Phi \Rightarrow (\Phi \vee (\exists(\Box\Psi)))$$

$$\forall \bigcirc \Phi \vee \Xi \Rightarrow \Psi \,\texttt{UNTIL}\, \Xi \quad \text{means} \quad ((\forall(\bigcirc\Phi)) \vee \Xi) \Rightarrow (\Psi \,\texttt{UNTIL}\, \Xi)$$

and so on...

# Example CTL formulae

Suppose `started` and `ready` are atomic propositions, then:

$$\exists\Diamond(\texttt{started} \wedge \neg\texttt{ready})$$

can be read as:

> *it is possible to get to a state where "started" holds but "ready" does not*

# Example CTL formulae

Suppose `started` and `ready` are atomic propositions, then:

$$\forall\Box\neg(\texttt{started} \land \neg\texttt{ready})$$

can be read as:

> *it is not possible to get to a state where "started" holds but "ready" does not*

# Example CTL formulae

Suppose `deadlock` is an atomic proposition, then:

$$\forall\Diamond\forall\Box\texttt{deadlock}$$

can be read as:

*the system will always progress to a state where it is henceforth permanently "deadlocked"*

# Example CTL formulae

Suppose `floor2`, `floor5`, `direction_up`, and `button_pressed_5` are atomic propositions, then:

$$\forall\Box(\texttt{floor2} \land \texttt{direction\_up} \land \texttt{button\_pressed\_5} \Rightarrow \forall(\texttt{direction\_up}\ \mathrm{UNTIL}\ \texttt{floor5}))$$

can be read as:

> *A lift on the second floor travelling upwards will always continue to travel upwards until reaching level 5 whenever it contains passengers wishing to reach that floor*

# Semantics of CTL

# Making intuition precise

Previous examples:

- Showed examples of properties expressible in CTL,
- Provided intuition for meaning of CTL formulae

Time to make that intuition precise...

# Models for CTL

Recall $\mathcal{M} = \langle S, S_0, \rightarrow, \mathcal{L} \rangle$, where:

- $S$ set of **states**
- $S_0 \subseteq S$ set of **initial states**
- $\rightarrow \subseteq S \times S$ (right-serial) **transition relation** on $S$
- $\mathcal{L} : S \rightarrow \mathbb{P}(AP)$ **labelling function**

"Right serial" means $\forall s \in S. \exists s' \in S. s \rightarrow s'$

# Infinite paths of states

Fix a CTL model $\mathcal{M} = \langle S, S_0, \rightarrow, \mathcal{L} \rangle$

Write $Paths(s)$ for set of infinite **paths** of $S$ starting at $s$

Write $\pi[i]$ for $i^{th}$ state of $\pi$ (“indexing”)

Write $\pi^i$ for suffix of $\pi$ starting position $i$

# Satisfaction at a state

Suppose $\mathcal{M}$ is a model, $s$ is a state in $\mathcal{M}$, and $\Phi$ is a state formula

Define the **satisfaction relation** $s \models \Phi$ recursively by:

$$
\begin{aligned}
s &\models \top && \text{always} \\
s &\models \bot && \text{never} \\
s &\models p && \text{iff } p \in \mathcal{L}(s) \\
s &\models \neg\Phi && \text{iff not } s \models \Phi
\end{aligned}
$$

# Satisfaction at a state

$$
\begin{aligned}
s &\models \Phi \vee \Psi && \text{iff } s \models \Phi \text{ or } s \models \Psi \\
s &\models \Phi \wedge \Psi && \text{iff } s \models \Phi \text{ and } s \models \Psi \\
s &\models \Phi \Rightarrow \Psi && \text{iff not } s \models \Phi \text{ or if } s \models \Phi \text{ and } s \models \Psi
\end{aligned}
$$

# Satisfaction at a state

$$s \models \forall\phi \qquad \text{iff } \pi \models \phi \text{ for every } \pi \in Paths(s)$$
$$s \models \exists\phi \qquad \text{iff } \pi \models \phi \text{ for some } \pi \in Paths(s)$$

$\pi \models \phi$ is the evaluation of path formula $\phi$ relative to a path $\pi$

# Satisfaction along a path

Suppose $\mathcal{M}$ is a model, $\pi$ is a path in $\mathcal{M}$, and $\phi$ is a path formula

Define the **satisfaction relation** $\pi \models \phi$ by:

$$
\begin{aligned}
&\pi \models \bigcirc\Phi && \text{iff } \pi[1] \models \Phi \\
&\pi \models \Box\Phi && \text{iff } \pi[i] \models \Phi \text{ for all } i \\
&\pi \models \Diamond\Phi && \text{iff } \pi[i] \models \Phi \text{ for some } i \\
&\pi \models \Phi \texttt{ UNTIL } \Psi && \text{iff } \pi[i] \models \Psi \text{ for some } i \text{ and } \pi[j] \models \Phi \text{ for all } j < i
\end{aligned}
$$

# Notes on satisfaction relations

Two relations are mutually recursive—mutually recursive grammar

Satisfaction relation for path formulae similar to LTL relation

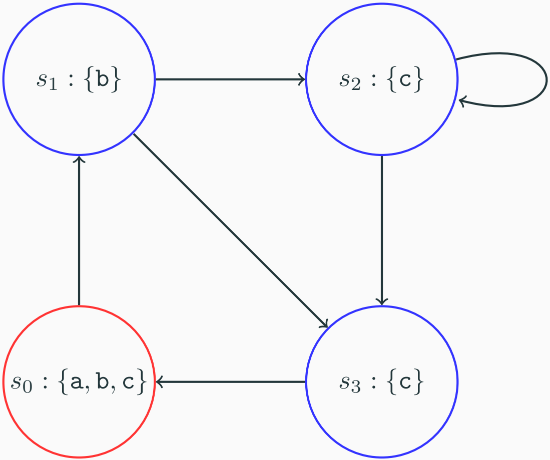
BUT:

- In LTL modality $\Box\phi$ uses all *suffixes* of path $\pi$
- In CTL modality $\Box\Phi$ uses all *indexes* of path $\pi$
- Similar for other modalities

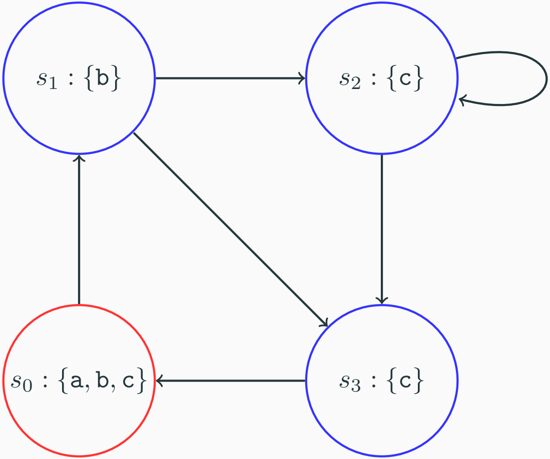
Tip: imagine types of $\pi^i$, $\pi[i]$ and satisfaction relations

# Examples

CTL model as a picture:

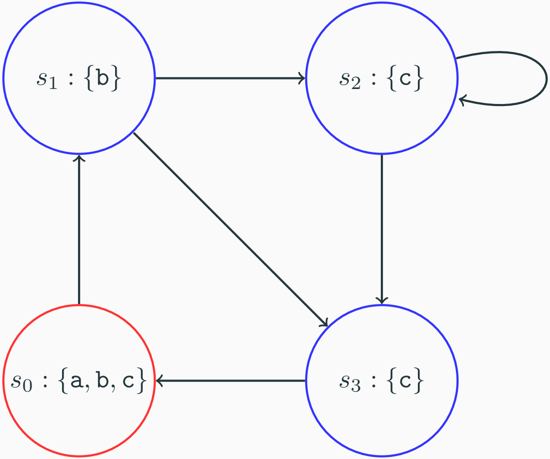


# Examples



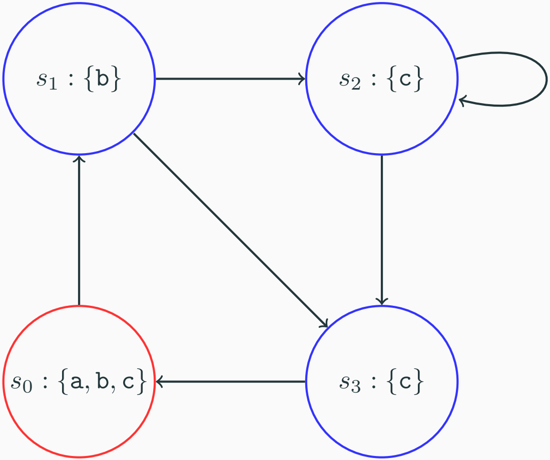

We have $s_0 \models \mathtt{a} \wedge \mathtt{b} \wedge \mathtt{c}$

# Examples



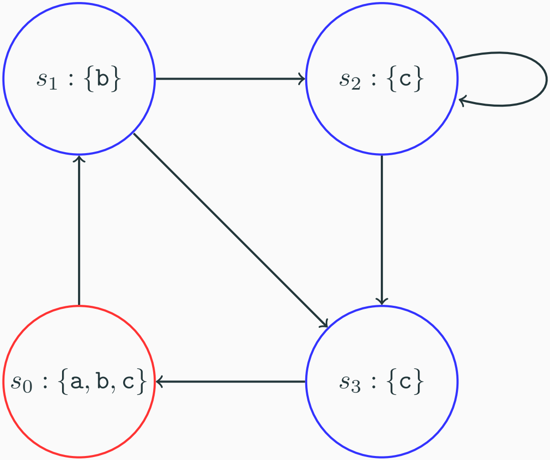

We have $s_0 \models \forall(\mathtt{b}\ \mathtt{UNTIL}\ \mathtt{c})$

# Examples



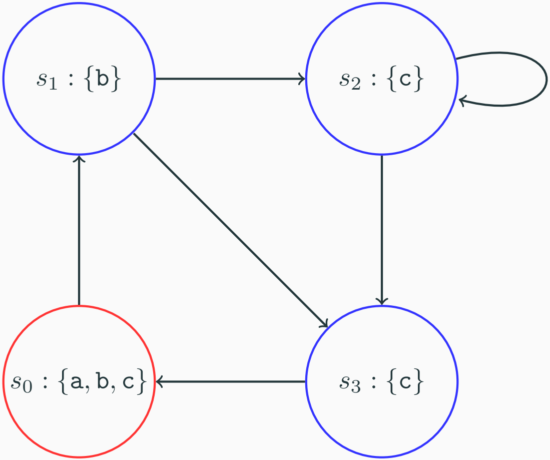

We have $s_1 \models \forall \bigcirc \mathtt{c}$

# Examples



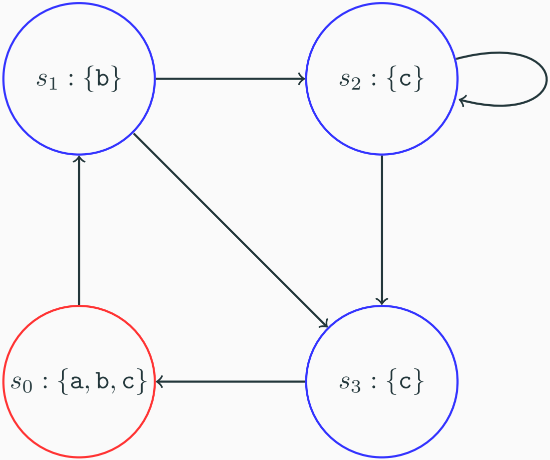

We have $s_1 \models \forall\bigcirc\forall\bigcirc\mathtt{c}$

# Examples




We have $s_1 \models \exists\Diamond\mathtt{a}$

# Examples




We have $s_2 \models \exists\Box\mathtt{c}$

# Semantic equivalence

# Satisfaction in model

Write $\mathcal{M} \models \Phi$ when $s \models \Phi$ for all states $s$ in $\mathcal{M}$

Read $\mathcal{M} \models \Phi$ as "model $\mathcal{M}$ satisfies $\Phi$"

Holds whenever all states of $\mathcal{M}$ satisfy $\Phi$

# Semantic equivalence

Say $\Phi$ and $\Psi$ are semantically equivalent ($\Phi \equiv \Psi$) when:

$$\mathcal{M} \models \Phi \text{ if and only if } \mathcal{M} \models \Psi \text{ for all models } \mathcal{M}$$

Intuitively $\Phi \equiv \Psi$ asserts that:

- $\Phi$ and $\Psi$ have same “semantic content”
- Safe to replace $\Phi$ with $\Psi$ (and vice versa) in any context
- Quantifying over $\mathcal{M}$ means can’t distinguish models

# Properties of semantic equivalence

Semantic equivalence:

- Is reflexive ($\phi \equiv \phi$)
- Is symmetric ($\phi \equiv \psi$ implies $\psi \equiv \phi$)
- Is transitive ($\phi \equiv \psi$ and $\psi \equiv \xi$ implies $\phi \equiv \xi$)

Also is congruent with structure of formulae

Example: $\phi_1 \equiv \phi_2$ implies $\neg\phi_1 \equiv \neg\phi_2$ and $\exists \bigcirc \phi_1 \equiv \exists \bigcirc \phi_2$

# Important semantic equivalences

$$\top \equiv \neg\bot$$

$$\Phi \Rightarrow \Psi \equiv \neg\Phi \vee \Psi$$

$$\Phi \vee \Psi \equiv \neg(\neg\Phi \wedge \neg\Psi)$$

# Important semantic equivalences

$$\forall \bigcirc \Phi \equiv \neg \exists \bigcirc \neg \Phi$$

$$\forall \Box \Phi \equiv \neg \exists (\top \texttt{ UNTIL } \neg \Phi)$$

$$\forall \Diamond \Phi \equiv \forall (\top \texttt{ UNTIL } \Phi)$$

$$\forall (\Phi \texttt{ UNTIL } \Psi) \equiv \neg \exists (\neg \Psi \texttt{ UNTIL } (\neg \Phi \wedge \neg \Psi)) \wedge \neg \exists \Box \neg \Psi$$

$$\exists \Diamond \Phi \equiv \exists (\top \texttt{ UNTIL } \Phi)$$

# Example proof

Task: show $\Phi \vee \Psi \equiv \neg(\neg\Phi \wedge \neg\Psi)$

Fix arbitrary model $\mathcal{M}$ and state $s$ in $\mathcal{M}$

Need to show $s \models \Phi \vee \Psi$ if and only if $s \models \neg(\neg\Phi \wedge \neg\Psi)$

# One direction

Assume $s \models \Phi \vee \Psi$

Then $s \models \Phi$ or $s \models \Psi$

Assume without loss of generality $s \models \Phi$

Then not $s \models \neg\Phi$

Hence not $s \models \neg\Phi \wedge \neg\Psi$

Therefore $s \models \neg(\neg\Phi \wedge \neg\Psi)$, as required

# T'other

Assume $s \models \neg(\neg\Phi \wedge \neg\Psi)$

Then not $s \models \neg\Phi$ and $s \models \neg\Psi$

Then not (not $s \models \Phi$ and not $s \models \Psi$)

Hence either $s \models \Phi$ or $s \models \Psi$

Without loss of generality, assume $s \models \Phi$

Then $s \models \Phi \vee \Psi$, as required

Therefore $\Phi \vee \Psi \equiv \neg(\neg\Phi \wedge \neg\Psi)$

# Existential Normal Form

Define formulae in Existential Normal Form (ENF) by:

$$\begin{aligned}\Phi, \Psi &::= \top \mid p \\ &::= \Phi \wedge \Psi \mid \neg \Phi \\ &::= \exists \bigcirc \Phi \mid \exists(\Phi \ \mathtt{UNTIL}\ \Psi) \mid \exists \Box \Phi\end{aligned}$$

Theorem:

*Every state formula has an equivalent ENF formula*

Proof: by structural induction, using previous semantic equivalences and congruences

Note proof is constructive: describes an algorithm

# Summary

- CTL uses a branching model of time
- CTL state formulae express “state properties” of systems
- CTL semantics with respect to states in model
- Equivalence when formulae have same “semantic content”
- Can use equivalences to rewrite a formula into ENF