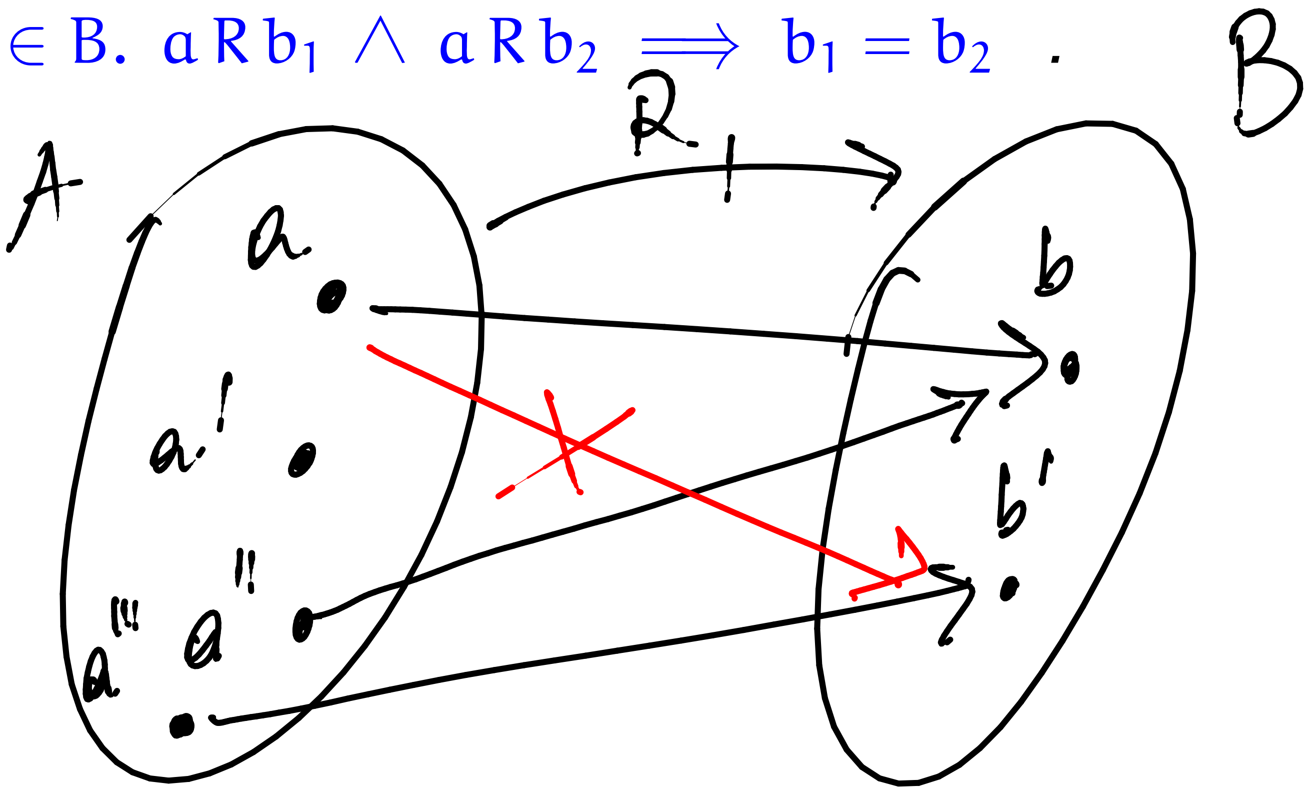
# Partial functions

**Definition 119** *A relation* $R : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ *is said to be* functional*, and called a* partial function*, whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B.\ a\,R\,b_1 \wedge a\,R\,b_2 \implies b_1 = b_2 \quad .$$

Idea: describes input-output behaviour.



⊛ it $f(a)$. In this case we write $f(a)\downarrow$:

**Theorem 121** *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

**NB**

$$f = g : A \rightharpoonup B$$

iff

$$\forall a \in A. \big( f(a)\downarrow \iff g(a)\downarrow \big) \wedge f(a) = g(a)$$

<u>Notation:</u> $f : A \rightharpoonup B$

$a \; f \; ?$

- There is no $b \in B$ s.t. $a \, f \, b$. We say $f(a)$ is undefined and write $f(a)\uparrow$
- there is a $b \in B$ s.t. $a \, f \, b$ and since it is unique depending on $a$, we call ⊛

**Example:** The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{N}$:

- for $n \geq 0$ and $m > 0$,
  $(n, m) \mapsto \big(\, \mathrm{quo}(n, m)\, ,\ \mathrm{rem}(n, m)\, \big)$
- for $n \geq 0$ and $m < 0$,
  $(n, m) \mapsto \big(\, -\mathrm{quo}(n, -m)\, ,\ \mathrm{rem}(n, -m)\, \big)$
- for $n < 0$ and $m > 0$,
  $(n, m) \mapsto \big(\, -\mathrm{quo}(-n, m) - 1\, ,\ \mathrm{rem}(m - \mathrm{rem}(-n, m), m)\, \big)$
- for $n < 0$ and $m < 0$,
  $(n, m) \mapsto \big(\, \mathrm{quo}(-n, -m) + 1\, ,\ \mathrm{rem}(-m - \mathrm{rem}(-n, -m), -m)\, \big)$

Its domain of definition is $\{\, (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \,\}$.

$(A \rightharpoonup B) \subseteq \mathrm{Rel}(A,B)$

**Proposition 122** *For all finite sets* $A$ *and* $B$,

$$\#(A \rightharpoonup B) \;=\; (\#B+1)^{\#A} \quad .$$

The set of all partial functions from $A$ to $B$.

PROOF IDEA:

$A = \{a_1, \ldots, a_n\} \qquad B = \{b_1, \ldots, b_m\}$

$$\begin{array}{lccccc}
\text{possible outputs} & b_m & b_m & \cdots & b_m \\
 & \vdots & \vdots & & \vdots \\
 & b_1 & b_1 & \cdots & b_1 \\
 & \uparrow & \uparrow & & \uparrow \\
\text{input} & a_1 & a_2 & \cdots & a_n
\end{array}
\qquad
\begin{array}{c}
\overbrace{(m+1)\times\cdots\times(m+1)}^{n \text{ times}} \\
\| \\
(m+1)^n
\end{array}$$

$(A \Rightarrow B) \subseteq (A \rightharpoonup B) \subseteq Rel(A,B)$

the set of all functions from A to B

# Functions (or maps)

**Definition 123** *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

$\forall a \in A.\ \exists b \in B.\ a\, f\, b$

**Theorem 124** *For all* $f \in \mathrm{Rel}(A, B)$,

by uniqueness we write $f(a)$ for such b.

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ a\, f\, b\ .$$

**Proposition 125** *For all finite sets* $A$ *and* $B$*,*

$$\#(A \Rightarrow B) = \#B^{\#A} \quad .$$

PROOF IDEA:

$A = \{a_1 \text{ —— } a_n\}$ $\qquad$ $B = \{b_1 \text{ —— } b_m\}$

possible outputs

$$\begin{array}{cccc} b_m & b_m & \cdots & b_m \\ \vdots & \vdots & & \vdots \\ b_2 & \vdots & & \vdots \\ b_1 & b_1 & \cdots & b_1 \\ \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_n \end{array}$$

$$\overbrace{m \times m \cdots \times m}^{n \text{ times}} = m^n$$

$\sim$ map to

**Theorem 126** *The identity partial function is a function, and the composition of functions yields a function.*

Principle of Extensionality

**NB**

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets $A$, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

   $$\mathrm{id}_A(a) = a$$

   and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

   $$(g \circ f)(a) = g\big(f(a)\big) \quad .$$

# Bijection

$\underline{Bij}(A, B) \subseteq (A \to B) \subseteq (A \rightharpoonup B) \subseteq \underline{Rel}(A, B)$

$f : A \to B$ is a bijection

Iff.

$\exists g : B \to A$ that is an inverse for $f$;

That is, $g \circ f = id_A$ $\quad \forall a \in A.\ g(f(a)) = a$

and $f \circ g = id_B$ $\quad \forall b \in B\ f(g(b)) = b$

Example: $\mathrm{Rel}([n]) \underset{\mathrm{rel}}{\overset{\mathrm{mat}}{\rightleftarrows}} (n \times n)\text{-matrices}$

Boolean

# Bijections

**Definition 127** *A function* $f : A \to B$ *is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function* $g : B \to A$ *(referred to as the* inverse *of* $f$*) such that*

1. $g$ *is a* retraction *(or* left inverse*) for* $f$*:*

$$g \circ f = \mathrm{id}_A \quad ,$$

2. $g$ *is a* section *(or* right inverse*) for* $f$*:*

$$f \circ g = \mathrm{id}_B \quad .$$

**Proposition 129** *For all finite sets* $A$ *and* $B$,

$$\#\,\mathrm{Bij}(A, B) = \begin{cases} 0 & \text{, if } \#A \neq \#B \\ n! & \text{, if } \#A = \#B = n \end{cases}$$

PROOF IDEA:

$A = \{a_1 \cdots a_n\}$ $\quad B = \{b_1 \cdots b_m\}$

$1 \leq i_1 \leq m$

$b_{i_2} \, (i_2 \neq i_1)$ $\quad i_3 \neq i_2, \; \neq i_1$

$b_{i_1} \quad \cancel{b_{i_1}} \quad b_{i_3} \quad \cdots$

$a_1 \quad a_2 \quad a_3 \cdots \quad a_n$

$m \times (m-1) \times (m-2) \times \cdots \times 1 = m!$

$m = n$

**Theorem 130** *The identity function is a bijection, and the composition of bijections yields a bijection.*

**Definition 131** *Two sets* $A$ *and* $B$ *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Enumerability

**Examples:**

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.
2. $\mathbb{N} \cong \mathbb{N}^{+}$ , $\mathbb{N} \cong \mathbb{Z}$ , $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$ , $\mathbb{N} \cong \mathbb{Q}$ . $(\mathbb{N} \not\cong \mathbb{R})$

$[\forall \text{ sets } S .\ S \not\cong \mathcal{P}(S)]$

# Equivalence relations and set partitions

- Equivalence relations.

$\underline{EqRel(A)} \subseteq \underline{Rel(A)}$

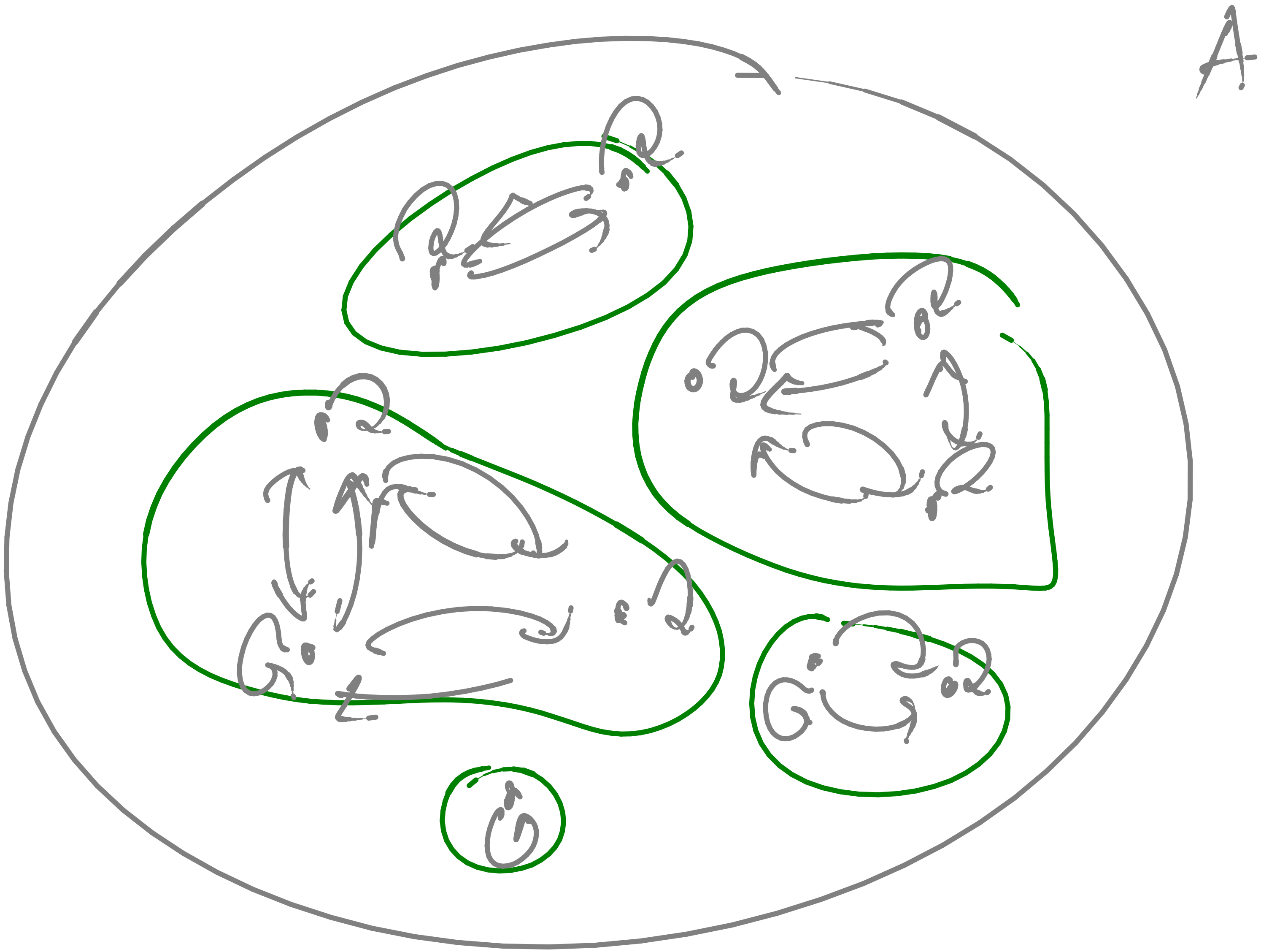
$\wr$

$E \subseteq A \times A$ s.t.

reflexive $\forall a \in A.\, a E a$

symmetry $\forall a, b \in A$
$a E b \Rightarrow b E a$

transitive $\forall a, b, c \in A.$
$a E b \wedge b E c$
$\Rightarrow a E c$




$$\underline{EqRel}(A) \cong \underline{Part}(A)$$

$\underline{\text{Part}}(A) = \{ \pi \mid \pi \text{ is a partition} \}$

$\pi \subseteq \mathcal{P}(A)$

$\forall c, c' \in \pi.\ c \neq c' \Rightarrow c \cap c' = \emptyset$

$\bigcup \{ c \mid c \in \pi \} = A$

$\forall c \in \pi.\ c \neq \emptyset.$