Recall: $s\, R^{\circ n}\, t$ iff $\exists$ path of length n from s to t in R.

$\{ S \in \mathrm{Rel}(A) \mid \exists n \in \mathbb{N}.\ S = R^{\circ n} \}$

Definition 114 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^{\circ n} .$$

Corollary 115 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s\, R^{\circ *}\, t$ *iff there exists a path with source* s *and target* t *in* R.

Consider A finite say of cardinality n:

$R^{\circ *} = \mathrm{Id} \cup R \cup R^{\circ 2} \cup \cdots \cup R^{\circ k} \cup \cdots =$

$(k \in \mathbb{N})$

<u>Claim</u> For $\#A = n$

$$R^{o*} = Id \cup R \cup \dots \cup R^{on}$$

(<u>NB</u>: A finite union)

Consider a path of length $k > n$

$$s = a_0 \, R \, a_1 \, R \, a_2 \, \cdots \cdots \quad a_n \cdots\cdots \quad a_{k-1} \, R \, a_k = t$$

Since $k > n$ $\exists$ $i \neq j$. s.t $a_i = a_j$

Then there is a path

$$s = a_0 \, R \, a_1 \, \cdots\cdots \, a_{i-1} \, R \, a_i = a_j \, R \, a_{j+1} \cdots\cdots R \, a_k = t$$

of shorter length.

We are interested in computing

$$M^* = I + M + M^2 + \cdots + M^n$$

for a matrix M.

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 &= I_n \\ M_{k+1} &= I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Preorders

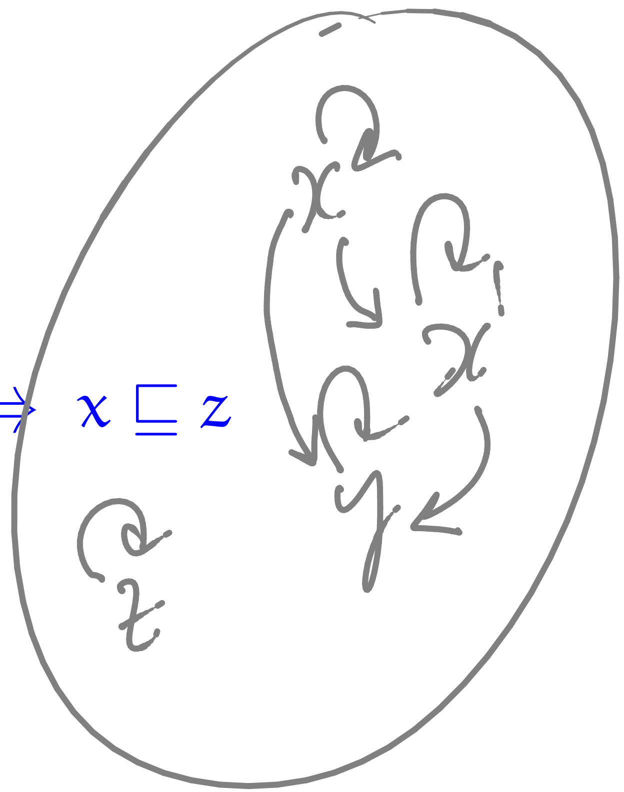
Definition 116 *A* preorder $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \wedge y \sqsubseteq z) \implies x \sqsubseteq z$$



N.B.: $R^{\circ *}$ is a preorder

RTP: $\forall x.\ x R^{0*} x$

iff $\exists$ a path from x to x

which is true because we always have the path of length 0.

RTP: $\forall x, y, z.\ x R^{0*} y \wedge y R^{0*} z \Rightarrow x R^{0*} z$

Let $x, y, z \in A$.

Assume $x R^{0*} y \Leftrightarrow \exists$ a path from x to y

$y R^{0*} z \Leftrightarrow \exists$ a path from y to z

We have, by concatenation of paths, a path from x to z. That is $x R^{0*} z$

$\boxtimes$

partial order $=$ preorder $+ \{$ antisymmetry

$x \leq y \wedge y \leq x \implies x = y$

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\ \mathbb{Z}\ ,\ |\ )$.

all the preorder that include the given relation R.

Theorem 118 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

E.g. $A \times A$ $\in$ $\mathcal{F}_R$

√ the least preorder that contains R.

PROOF:

Idea: $a (\bigcap \mathcal{F}_R) b \quad \forall a R b$

$a (\bigcap \mathcal{F}_R) a \quad \forall a$

$a R b \wedge b R c$
$\Rightarrow a (\bigcap \mathcal{F}_R) b \wedge b (\bigcap \mathcal{F}_R) c$
$\Rightarrow a (\bigcap \mathcal{F}_R) c$

$c R d \Rightarrow c (\bigcap \mathcal{F}_R) d \Big\} \Rightarrow a (\bigcap \mathcal{F}_R) d$

RTD: $\bigcup_{n\in\mathbb{N}} R^{on} \subseteq \bigcap \mathcal{F}_R$

$\Updownarrow$

$\forall n\in\mathbb{N}.\ R^{on} \subseteq \bigcap \mathcal{F}_R$

$\Updownarrow$

$\forall n\in\mathbb{N}.\ \forall Q$ a prendur s.t $R\subseteq Q.$

$R^{on} \subseteq Q$

We prove it by induction.

Lemma.

$\left(\bigcup_{i\in I} X_i\right) \subseteq Y$

$\Leftrightarrow$

$\forall i\in I.$

$X_i \subseteq Y$

$X \subseteq \bigcap_{j\in J} Y_j$

$\Leftrightarrow$

$\forall j\in J.$

$X \subseteq Y_j$

RTP:

<u>Base case</u>: $\forall Q$ preorder s.t. $Q \supseteq R$

$$Id \subseteq Q.$$

$$\Updownarrow$$

$$\forall x,y.\ x\,Id\,y \Rightarrow x\,Q\,y$$

$$\Updownarrow$$

$$\forall x,y.\ x=y \Rightarrow x\,Q\,y$$

$$\Updownarrow$$

$$\forall x.\ x\,Q\,x$$

true because Q is reflexive.

Inductive Step

Lemma $X \subseteq X'$, $Y \subseteq Y'$ $\Rightarrow X \circ Y \subseteq X' \circ Y'$

(IH) Assume: $\forall Q$ preorders s.t. $R \subseteq Q$. $R^{\circ n} \subseteq Q$

RTP: $\forall Q'$ preorders s.t. $R \subseteq Q'$. $R^{\circ n+1} \subseteq Q'$

Let Q' be a preorder s.t. $R \subseteq Q'$.

RTP $R^{\circ n+1} \subseteq Q'$

$$R^{\circ n+1} = R \circ R^{\circ n} \overset{?}{\subseteq} Q'$$

By (IH): $R^{\circ n} \subseteq Q'$

$\Rightarrow R \circ R^{\circ n} \subseteq Q' \circ Q'$

Lemma $Q^!$ a prerder $\Rightarrow$ $Q^! \circ Q^! \subseteq Q^!$

Then

$$\underbrace{R \circ R^{\circ n}}_{= R^{\circ n+1}} \subseteq Q^! \circ Q^! \subseteq Q^!$$

So $R^{\circ n+1} \subseteq Q^!$.

$\boxtimes$