$R : A \mapsto B \underset{\text{def}}{\sim} R \subseteq A \times B$

Relational extensionality

$$R = S : A \twoheadrightarrow B$$

iff

$$\forall a \in A.\, \forall b \in B.\ a\,R\,b \iff a\,S\,b$$

Relational composition

$A \overset{R}{\longrightarrow\!\!\!\!\!\!+\;} B \quad B \overset{S}{\longrightarrow\!\!\!\!\!\!+\;} C \quad \rightsquigarrow \quad A \overset{S \circ R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} C$

$(a, c) \in (S \circ R) \overset{\text{def}}{\iff} \exists\, b \in B \,.\, (b, c) \in S \wedge (a, b) \in R$

$\left[\; a\,(S \circ R)\,c \overset{\text{def}}{\iff} \exists\, b \in B \,.\; b\,S\,c \wedge a\,R\,b \,.\; \right]$

$(a,d) \in ((T \circ S) \circ R) \iff (a,d) \in (T \circ (S \circ R))$

Exercise

Theorem 102 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

  For all $R : A \nrightarrow B$, $S : B \nrightarrow C$, *and* $T : C \nrightarrow D$,

  $$\underbrace{(T \circ S)}_{B \nrightarrow C} \circ R \;=\; T \circ \underbrace{(S \circ R)}_{A \nrightarrow C} \quad : A \nrightarrow D$$

  It is unambiguous to write $T \circ S \circ R$

- *Neutral element.*

  For all $R : A \nrightarrow B$,

  $$R \circ \mathrm{id}_A \;=\; R \;=\; \mathrm{id}_B \circ R \quad .$$

$\mathrm{id}_X : X \nrightarrow X$

$(x, x') \in \mathrm{id}_X \overset{\text{def}}{\iff} x = x'$

Definition 103

1. *For positive integers* m *and* n*, an* $(m \times n)$-<u>matrix</u> M *over a semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$.

$$M = \begin{array}{cc} & \begin{array}{ccccc} 0 & \cdots & i & \cdots & m-1 \end{array} \\ \begin{array}{c} 0 \\ \vdots \\ j \\ \vdots \\ n-1 \end{array} & \left[\begin{array}{ccccc} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & M_{ij} & & \\ & & & & \\ & & & & \end{array}\right] \end{array}$$

Theorem 104 *Matrix multiplication is associative and has the identity matrix as neutral element.*

I $(k \times k)$-matrix $\quad I_{i,j} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

M $(m \times n)$-matrix

L $(l \times m)$-matrix $\rightsquigarrow$ $M \cdot L$ $(l \times n)$-matrix

$$(M \cdot L)_{i,j} = \bigoplus_{k=0}^{m-1} M_{k,j} \odot L_{i,k}$$

Consider the semiring of Booleans $(\{0,1\}, \overset{\vee}{+}, \overset{\wedge}{\cdot}, 0, 1)$

$$(M \cdot L)_{i,j} = \bigvee_{k=0}^{m-1} M_{k,j} \wedge L_{i,k} \iff \exists k.\ M_{k,j} \wedge L_{i,k}$$

$(m \times n)$-matrices over Booleans.

$[m] \xrightarrow{} [n]$

$[k] = \{0, 1, \ldots, k-1\}$

$M \rightsquigarrow \underline{\mathrm{rel}}(M)$

$\underline{\mathrm{Def}}$ $(i,j) \in \underline{\mathrm{rel}}(M) \iff M_{i,j} = 1$

$\underline{\mathrm{mat}}(R) \leftsquigarrow R$

$$\underline{\mathrm{Def}} \quad \underline{\mathrm{mat}}(R)_{i,j} = \begin{cases} 0 & (i,j) \notin R \\ 1 & (i,j) \in R \end{cases}$$

Exercises

$$M \longrightarrow \mathrm{rel}(M) \longrightarrow \mathrm{mat}(\mathrm{rel}(M))$$

$$M \overset{(*)}{=} \mathrm{mat}(\mathrm{rel}(M))$$

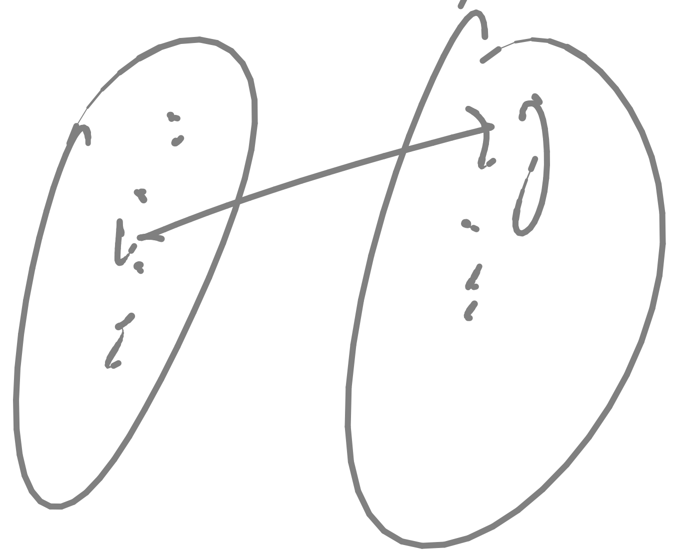
$$R \longrightarrow \mathrm{mat}(R) \longrightarrow \mathrm{rel}(\mathrm{mat}(R))$$

$$R \overset{(*)}{=} \mathrm{rel}(\mathrm{mat}(R))$$

$$M = \begin{bmatrix} \ddots & & \cdot^{\cdot^{\cdot}} \\ & M_{ij} & \\ \cdot^{\cdot^{\cdot}} & & \ddots \end{bmatrix}$$

$M_{ij} = 1$

$\mathrm{rel}(M)$



$$\mathrm{mat}(\mathrm{rel}(M)) = \begin{array}{c c} & i \\ j & \begin{bmatrix} & \vdots & \\ - & 1 & \\ & & \end{bmatrix} \end{array}$$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

M $(m \times n)$-mat $\rightsquigarrow$ $\underline{\text{rel}}(M) : [m] \nrightarrow [n]$

L $(l \times m)$-mat $\rightsquigarrow$ $\underline{\text{rel}}(L) : [l] \nrightarrow [m]$

$M \bullet L$ $(l \times n)$-mat

$\underline{\text{rel}}(M) \circ \underline{\text{rel}}(L) : [l] \nrightarrow [n]$

$\parallel$

$\underline{\text{rel}}(M \bullet L) : [l] \nrightarrow [n]$

<u>Prop</u>

Directed graphs

$R \circ R$,
$R \circ R \circ R$, ...

Definition 108 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*

$R : A \nrightarrow A$

$R \subseteq A \times A$

$(a, a') \in R$

$(a, a'') \in R$

$\vdots$

$= \mathcal{P}(A \times A)$

$n=0$

$n=1$

$n=2$

$a(R \circ R)b$

Corollary 110 *For every set* A*, the structure*

$$(\ \mathrm{Rel}(A)\ ,\ \mathrm{id}_A\ , \circ\ )$$

is a monoid. $\boxed{?}$ $a(R^{\circ n})b \iff \ldots$?

Definition 111 *For* $R \in \mathrm{Rel}(A)$ *and* $n \in \mathbb{N}$*, we let*

$$R^{\circ n} = \underbrace{R \circ \cdots \circ R}_{n \textit{ times}} \in \mathrm{Rel}(A)$$

be defined as id_A *for* $n = 0$*, and as* $R \circ R^{\circ m}$ *for* $n = m + 1$.

Proposition 113 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s\ R^{\circ n}\ t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF: By induction on,

Base case $(n=0)$ $\quad s\, R^{\circ 0}\, t \overset{?}{\iff} \exists$ path of length 0 from s to t

$R^{\circ 0} = \mathrm{id}$ $\quad \Updownarrow \quad\quad \Updownarrow$

$s = t \quad\quad s = t$

Inductive step: $(n = m+1)$

(IH) Assume $s\, R^{\circ m}\, t \iff \exists$ path of length m from s to t

RTP $s\, R^{\circ m+1}\, t \overset{?}{\iff} \exists$ path of length $m+1$ from s to t.

$$\Updownarrow$$

$$s\, (R \circ R^{\circ m})\, t$$

$$\Updownarrow$$

$\exists p. (p\, R^{\circ m}\, t) \wedge (s\, R\, p) \overset{(IH)}{\iff} \exists$ path of length m from p to t $\wedge$ $s\, R\, p$

$\boxtimes$

Definition 114 *For* $R \in \mathrm{Rel}(A)$*, let*

$$R^{\circ *} \;=\; \bigcup \left\{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \right\} \;=\; \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 115 *Let* (A, R) *be a directed graph. For all* $s, t \in A$*,* $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R*.*