Ordered pairing

For every pair a and b, the set

$$\{\,\{\,a\,\}\,,\,\{\,a,b\,\}\,\}$$

is abbreviated as

$$\langle a,b\rangle$$

and referred to as an *ordered pair*.

Proposition 87 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff (a = x \wedge b = y) \quad .$$

PROOF: Let a, b, x, y be arbitrary.

$(\Leftarrow)$ Easy

$(\Rightarrow)$ Assume $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$.

Case 1: $a = b$

Then $\langle a, b\rangle = \{\{a\}\} = \{\{x\}, \{x, y\}\}$

$\Downarrow$

$\{x\} = \{x, y\}$

$\Downarrow$

$x = y$

Thus, $\{\{a\}\} = \{\{x\}\} \Rightarrow \{a\} = \{x\} \Rightarrow a = x$

$\Rightarrow b = y$

Case $a \neq b$: Recall assumption

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$$

$\#\langle a,b\rangle = 2$ so $\#\langle x,y\rangle = 2 \Rightarrow x \neq y$

Case: $\{a\} = \{x\} \Rightarrow a = x$

Thus $\{\{a\},\{a,b\}\} = \{\{a\},\{a,y\}\}$

$\Rightarrow \{a,b\} = \{a,y\} \Rightarrow b = y$

Case $\{a\} = \{x,y\}$ is impossible

because $\#\{a\} = 1$ and $\#\{x,y\} = 2$

$\boxtimes$

like $\alpha * \beta$ in ML.

an ordered pair construction

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{ x \mid \exists a \in A, b \in B.\, x = (a, b) \}$$

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B.$$
$$(a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \; .$$

Thus,

$$\forall x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \; .$$

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$$

Proposition 89 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

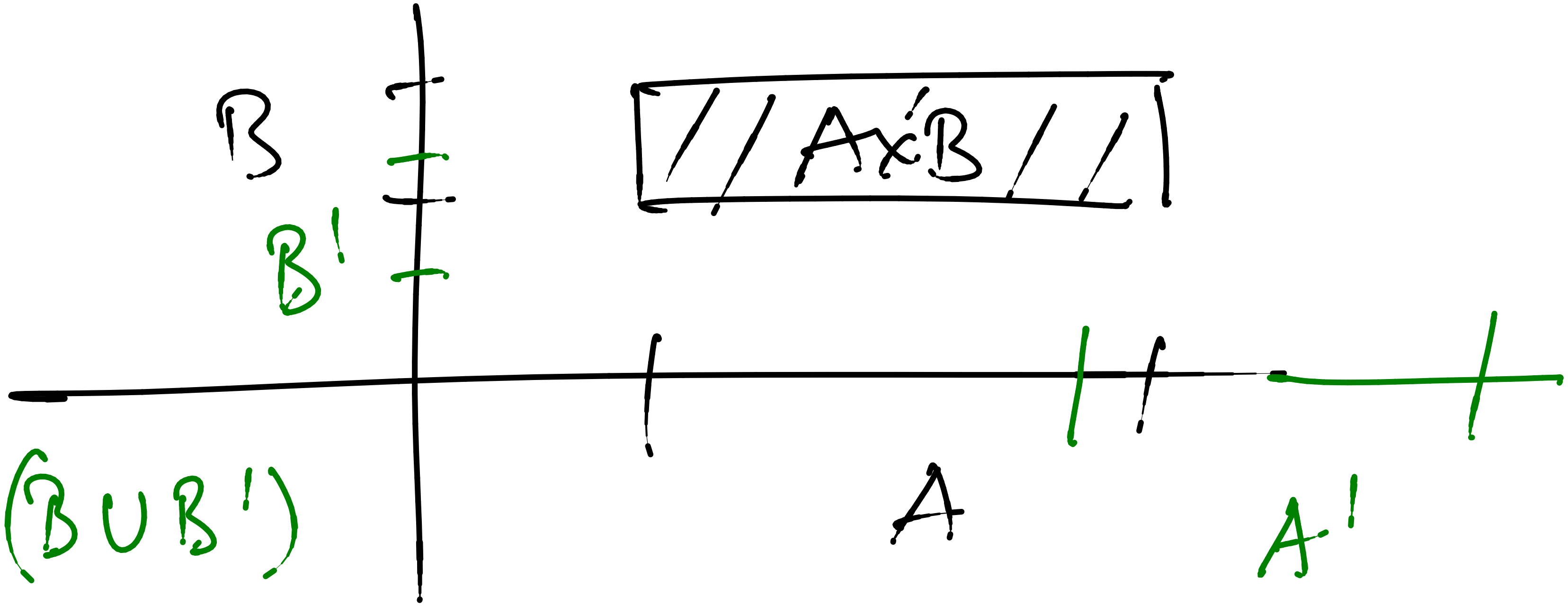
$A \times B$

PROOF IDEA:

$$
B \quad
\begin{array}{lccccc}
b_m & \{(a_1,b_m), & \ldots, & (a_i,b_m), & \ldots, & (a_n,b_m), \\
\vdots & \vdots & & \vdots & & \vdots \\
b_j & (a_1,b_j), & \ldots, & (a_i,b_j), & \ldots, & (a_n,b_j), \\
\vdots & & & & & \vdots \\
b_2 & \vdots & & \vdots & & \\
b_1 & (a_1,b_1), & \ldots, & (a_i,b_1) & \ldots, & (a_n,b_1)\} \\
 & a_1 \quad a_2 & \ldots & a_i & \ldots & a_n
\end{array}
$$

A

Intuitively:



$(A \cup A') \times (B \cup B')$

vs.

$(A \times B) \cup (A' \times B')$

Big unions

Definition 90 *Let* $\mathbb{U}$ *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathbb{U}))$, *we let the* big union *(relative to* $\mathbb{U}$*) be defined as*

Cf:

$$\bigcup \mathcal{F} = \{ x \in \mathbb{U} \mid \exists A \in \mathcal{F}. x \in A \} \in \mathcal{P}(\mathbb{U}) \quad .$$

flat : α list list $\to$ α list

$[\ell_1, \ldots, \ell_n] \mapsto (\ell_1 @ \cdots @ \ell_n)$

$\mathcal{F}$ is a set of sets (of $\mathcal{U}$)

$\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$

intuitively ||

Idea:

$\bigcup \mathcal{F} = \cdots \cup A \cup \cdots$

$\{ \ldots, A, \ldots \}$ with $A \subseteq \mathcal{U}$

Proposition 91 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{U})))$,

$$\bigcup\Big(\bigcup\mathcal{F}\Big) = \bigcup\underbrace{\Big\{\bigcup\mathcal{A} \in \mathcal{P}(\mathbb{U}) \;\Big|\; \mathcal{A} \in \mathcal{F}\Big\}}_{\text{formally } = \; \{X \in \mathcal{P}U \mid \exists \mathcal{A} \in \mathcal{F}.\, X = \bigcup \mathcal{A}\}} \in \mathcal{P}(\mathbb{U}) \quad .$$

PROOF:

$\mathcal{P}\mathcal{P}U$

$$\mathcal{F} = \{ \ldots, \mathcal{A}, \ldots, \mathcal{A}', \ldots \}$$

$\mathcal{A} \in \mathcal{P}\mathcal{P}U$

$\mathcal{A}' = \{ \ldots X', \ldots Y', Z', \ldots \}$

$\mathcal{A} = \{ \ldots, X, \ldots, Y, \ldots, Z, \ldots \}$

$X, Y, Z \ldots \subseteq U$

$$\bigcup \mathcal{F} = \{ \ldots, X, \ldots, Y, \ldots Z, \ldots X', \ldots Y', \ldots Z' \ldots \}$$

$$\bigcup\Big(\bigcup \mathcal{F}\Big) = \ldots \cup X \cup \ldots \cup Y \cup \ldots \cup Z \cup \ldots \cup X' \cup \ldots$$

$$\mathcal{F} = \{ \dots, \mathcal{A}, \dots, \mathcal{A}', \dots \}, \quad \mathcal{A}' = \{ \dots X', \dots Y', \dots Z' \dots \}$$

$$\mathcal{A} = \{ \dots, X, \dots, Y, \dots Z, \dots \}$$

$$\bigcup \mathcal{A} = (\dots \cup X \cup \dots \cup Y \cup \dots \cup Z \cup \dots)$$

$$\bigcup \mathcal{A}' = (\dots \cup X' \cup \dots \cup Y' \cup \dots \cup Z' \cup \dots)$$

$$\bigcup \{ \bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F} \}$$

$$= \dots \cup (\dots \cup X \dots \cup Y \dots \cup Z) \cup \dots (\dots \cup X' \dots \cup Y' \dots)$$

$$\cup \dots$$

IDEA: $\bigcup(\bigcup \mathcal{F}) = \bigcup\{\bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F}\}$

$(\subseteq)$ $x \in \bigcup(\bigcup \mathcal{F})$

RTP: $x \in \bigcup\{\bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F}\}$

$\exists X \in \bigcup \mathcal{F}.\ x \in X$

$\iff \exists X.\ X \in \bigcup \mathcal{F} \wedge x \in X$

$(\supseteq)$

$\iff \exists X.\ \exists A \in \mathcal{F}.\ X \in A \wedge x \in X$

$x \in \bigcup\{\bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F}\}$

$\iff \exists S.\ \exists \mathcal{A} \in \mathcal{F}.\ S = \bigcup \mathcal{A} \wedge x \in S$