We can collect all subsets of a given set, say U, into a new set which is called the <u>powerset</u> and denoted $\mathcal{P}(U)$.

Powerset axiom

For any set, there is a set consisting of all its subsets.

$$\mathcal{P}(U)$$

Recall

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U\ .$$

$$A \subseteq B \iff (\forall x.\ x \in A \Rightarrow x \in B) \iff \forall x \in A.\ x \in B$$

The powerset construction increases cardinality

$\# \emptyset = 0$

$\# \mathcal{P}(\emptyset) = 2^0 = 1$

$\# PP\emptyset = 2^1 = 2$

$\vdots$

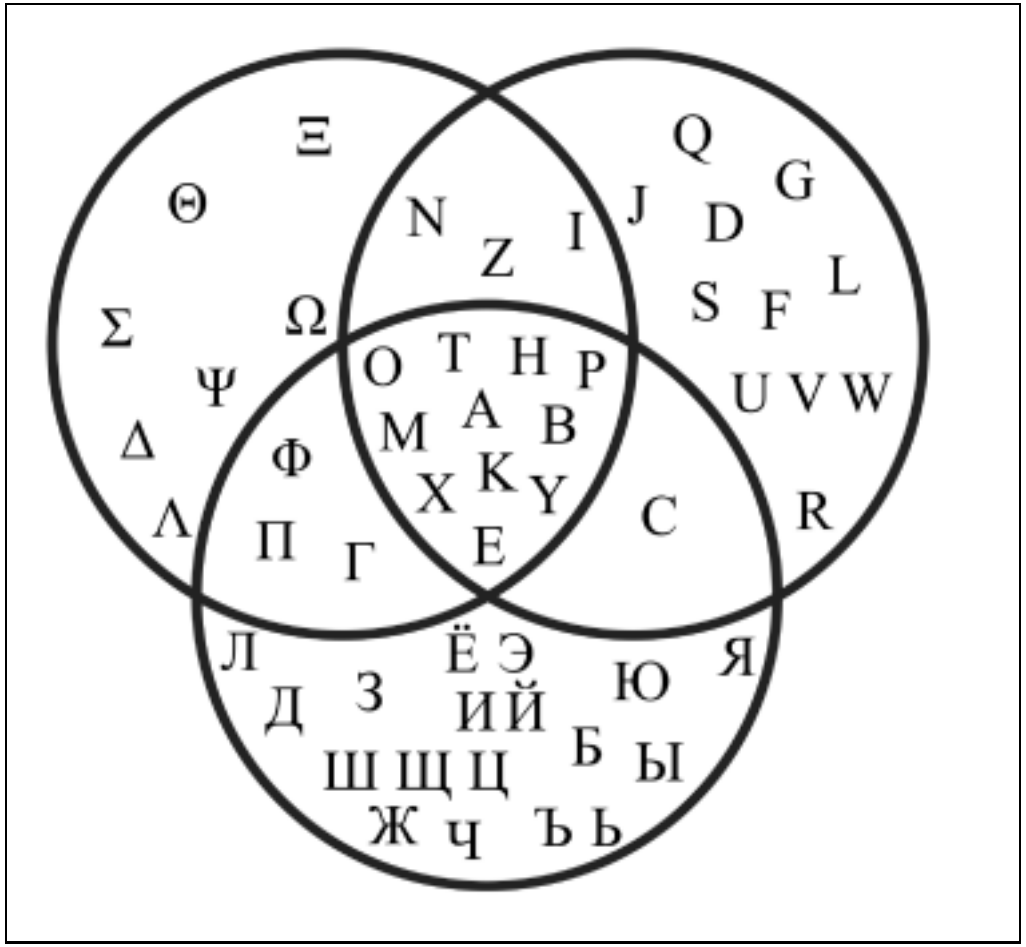
$\# \underbrace{PP \cdots P}_{n+1 \text{ times}}(\emptyset) = 2^n$.

<u>Recall</u>

$\# A = a$

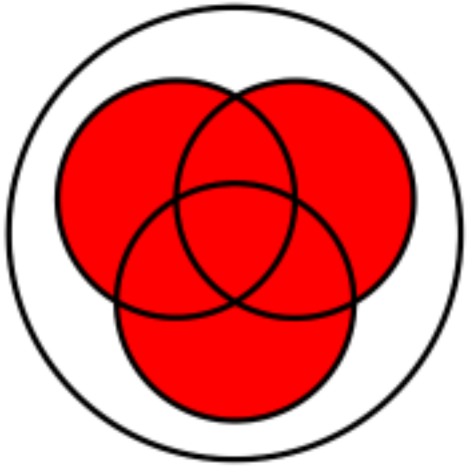
$\Rightarrow \# PA = 2^a$

Venn diagrams[a]

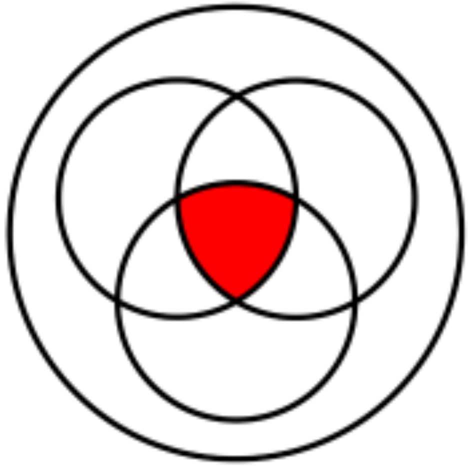


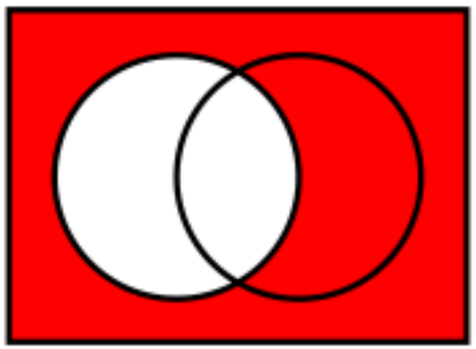
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

The powerset Boolean algebra

$$\left(\ \mathcal{P}(U)\ ,\ \emptyset\ ,\ U\ ,\ \cup\ ,\ \cap\ ,\ (\cdot)^c\ \right)$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\} \in \mathcal{P}(U)$$

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\} \in \mathcal{P}(U)$$

$$A^c = \{x \in U \mid \neg(x \in A)\} \in \mathcal{P}(U)$$

notation: $x \notin A$

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{$ **false** , **true** $\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$

Proposition 85 *Let* U *be a set and let* $A, B \in \mathcal{P}(U)$.

1. $\forall X \in \mathcal{P}(U).\ A \cup B \subseteq X \iff (A \subseteq X \land B \subseteq X)$.
2. $\forall X \in \mathcal{P}(U).\ X \subseteq A \cap B \iff (X \subseteq A \land X \subseteq B)$.

Exercise

PROOF: Let $X \in \mathcal{P}(U)$; that is, $X \subseteq U$.

$(\Rightarrow)$ Assume $A \cup B \subseteq X$

$A \subseteq A \cup B$

$A \subseteq X$

Analogously one shows $B \subseteq X$.

$(\Leftarrow)$ Assume $A \subseteq X$ and $B \subseteq X$.

Lemma: $\forall A.\ A \subseteq A \cup B$

Lemma: $A \subseteq B \land B \subseteq C \Rightarrow A \subseteq C$

RTP $A \cup B \subseteq X$

that is, $\forall x.\ (x \in A) \vee (x \in B) \Rightarrow x \in X$

Assume $x \in \mathcal{U}$ s.t. $(x \in A) \vee (x \in B)$

RTP $x \in X$

Case (1) $x \in A$ and so, since $A \subseteq X$, $x \in X$.

Case (2) $x \in B$ then $x \in X$ because $B \subseteq X$

$\boxtimes$

Corollary 86 *Let* U *be a set and let* $A, B, C \in \mathcal{P}(U)$.

Proof principle for showing a set is a union or an intersection.

1. $$C = A \cup B$$

   iff

   $$[A \subseteq C \wedge B \subseteq C]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(U).\ (A \subseteq X \ \wedge\ B \subseteq X) \implies C \subseteq X]$$

2. $$C = A \cap B$$

   iff

   $$[C \subseteq A \wedge C \subseteq B]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(U).\ (X \subseteq A \ \wedge\ X \subseteq B) \implies X \subseteq C]$$

The Laws of Boolean Algebras.

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A = A = \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A = A \cap (A \cup B)$$

- The complement operation $(\cdot)^c$ satisfies complementation laws.

$$A \cup A^c = \mathcal{U} \;, \quad A \cap A^c = \emptyset$$

<u>Exercise</u> Prove the De Morgan laws:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Unordered

Pairing axiom

$\{a,b\} = \{b,a\}$

For every a and b, there is a set with a and b as its only elements.

$$\{a\,,\,b\} \qquad x \in \{b,a\}$$

$$\Updownarrow$$

defined by

$$\Leftarrow\!\Rightarrow (x = b \vee x = a)$$

$$\forall x.\, \underline{x \in \{a,b\}} \iff (x = a \vee x = b)$$

NB The set $\{a,a\}$ is abbreviated as $\{a\}$, and referred to as a *singleton*.

$$x \in \{a,a\} \Leftrightarrow (x = a \vee x = a) \Leftrightarrow (x = a)$$

Examples:

$\{\emptyset\} \neq \{\{\emptyset\}\}$

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Exercise $\langle a,b\rangle = \langle x,y\rangle \iff a=x \wedge b=y$

(Cor: $\langle a,b\rangle = \langle b,a\rangle \iff a=b$)

Ordered pairing

For every pair a and b, the set

$$\{\,\{\,\mathrm{a}\,\}\,,\,\{\,\mathrm{a},\mathrm{b}\,\}\,\}$$

is abbreviated as

$$\langle \mathrm{a},\mathrm{b}\rangle \quad \neq \quad \langle b,a\rangle$$

not necessarily the same.

and referred to as an *ordered pair*.

E.g. $\langle 1,2\rangle = \{\{1\},\{1,2\}\}$ $\times$

$\langle 2,1\rangle = \{\{2\},\{2,1\}\}$