Denotational semantics of PCF terms, IV

$$\begin{bmatrix} \Gamma \vdash \mathbf{fn} \, x : \tau \, . \, M \end{bmatrix} (\rho) \\ \stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket \, . \, \llbracket \Gamma[x \mapsto \tau] \vdash M \rrbracket (\rho[x \mapsto d]) \qquad (x \notin dom(\Gamma))$$

NB: $\rho[x \mapsto d] \in \llbracket \Gamma[x \mapsto \tau] \rrbracket$ is the function mapping x to $d \in \llbracket \tau \rrbracket$ and otherwise acting like ρ .

(*) $f: D_1 \times D_2 \rightarrow D$ cont. chim $\hat{f}: D_1 \rightarrow (D_2 \rightarrow D)$ cont. $\hat{f}(d_1) = \lambda d \cdot f(d_1, d)$ Assume (X) f is nonstone. $d_{1} \equiv d_{i} \xrightarrow{?} \hat{f}(d_{1}) \equiv \hat{f}(d_{i})$ $(d_{1}, d) = (d_{1}, d)$ if $\widehat{f}(d_{1})(d) = \widehat{f}(d_{1})(d)$ if $(d_{1}) = \widehat{f}(d_{1})(d)$ if d $\int f(d_1, d) = f(d_1, d)$ fmsh.

• 7 previves hubs $\int f(\Box_n dn) \stackrel{?}{=} \Box_n f(dn)$ $= \iint f(\Box_n dn) (d) \stackrel{?}{=} (\Box_n f(dn)) (d)$ $f(U_nd_n,d)$ $U_n(f(d_n)(d))$ f cont (in first arg.) $\Box_n f(dn, d)$

Denotational semantics of PCF terms, V

$$\llbracket \Gamma \vdash \mathbf{fix}(M) \rrbracket(\rho) \stackrel{\text{def}}{=} fix(\llbracket \Gamma \vdash M \rrbracket(\rho))$$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. For all typing judgements $\Gamma \vdash M : \tau$, the denotation

$\llbracket \Gamma \vdash M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_{\tau}$, we get

 $\llbracket \emptyset \vdash M \rrbracket : \llbracket \emptyset \rrbracket \to \llbracket \tau \rrbracket$

and, since $\llbracket \emptyset \rrbracket = \{ \bot \}$, we have

 $\llbracket M \rrbracket \stackrel{\text{def}}{=} \llbracket \emptyset \vdash M \rrbracket (\bot) \in \llbracket \tau \rrbracket \qquad (M \in \mathrm{PCF}_{\tau})$

Compositionality

Proposition. For all typing judgements $\Gamma \vdash M : \tau$ and $\Gamma \vdash M' : \tau$, and all contexts $\mathcal{C}[-]$ such that $\Gamma' \vdash \mathcal{C}[M] : \tau'$ and $\Gamma' \vdash \mathcal{C}[M'] : \tau'$, if $[\Gamma \vdash M] = [\Gamma \vdash M'] : [\Gamma] \rightarrow [\tau]$

then $\llbracket \Gamma' \vdash \mathcal{C}[M] \rrbracket = \llbracket \Gamma' \vdash \mathcal{C}[M] \rrbracket : \llbracket \Gamma' \rrbracket \rightarrow \llbracket \tau' \rrbracket$

•
$$M = f_{\infty}(M!)$$

 $M = f_{\infty}(M!)$
 $M = f_{\infty}(M!)$

 $M = M_1 M_2$ $M[\frac{n^2}{2}] \downarrow V$ MIL forz. M $M_1 M_2 \cup V$ by nd. [[M]=[[fnx. M]=]d. [[M][xnd] emma ([M[M2/2]]).AV] $\Pi M, M_2 \mathcal{Y} \doteq \Pi \mathcal{V} \mathcal{Y}$ KTP: $[Im, Y(Im_2Y) \rightarrow [ImY] (2H) [2H) [2H)$

Proposition. Suppose that $\Gamma \vdash M : \tau$ and that $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, so that we also have $\Gamma \vdash M'[M/x] : \tau'$. *Then*,

$$\begin{bmatrix} \Gamma \vdash M'[M/x] \end{bmatrix} (\rho) \\ = \begin{bmatrix} \Gamma[x \mapsto \tau] \vdash M' \end{bmatrix} (\rho[x \mapsto \llbracket \Gamma \vdash M] \end{bmatrix})$$

for all $\rho \in \llbracket \Gamma \rrbracket$.

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau \rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and $[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau \rangle \vdash M']\!] ([\![M]\!])$

Topic 7

Relating Denotational and Operational Semantics

For any closed PCF terms M and V of ground type $\gamma \in \{nat, bool\}$ with V a value

$$\llbracket M \rrbracket = \llbracket V \rrbracket \in \llbracket \gamma \rrbracket \implies M \Downarrow_{\gamma} V.$$

NB. Adequacy does not hold at function types:

$$\llbracket \mathbf{fn} \ x : \tau. \ (\mathbf{fn} \ y : \tau. \ y) \ x \rrbracket = \llbracket \mathbf{fn} \ x : \tau. \ x \rrbracket \quad : \llbracket \tau \rrbracket \to \llbracket \tau \rrbracket$$

but

 $\mathbf{fn} \ x:\tau. \left(\mathbf{fn} \ y:\tau. \ y\right) x \not \downarrow_{\tau \to \tau} \mathbf{fn} \ x:\tau. \ x$

Adequacy proof idea

M, M, DV

- 1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.
 - Consider M to be $M_1 M_2$, fix(M').

• $M = M_1 M_2$

 $[[M]] = [[V]] =) M U_{n} V$

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

• Consider M to be $M_1 M_2$, $\mathbf{fix}(M')$.

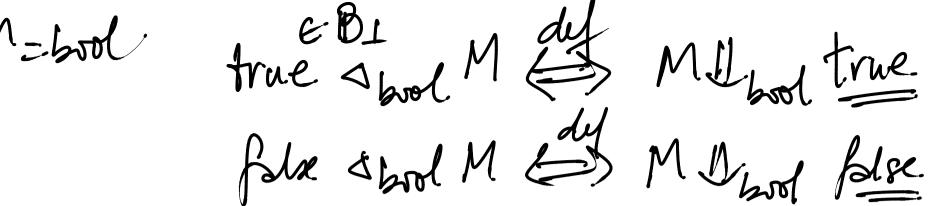
2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy. If T a grad type This statement roughly takes the form: M = M for all types τ and all $M \in PCF_{\tau}$

where the *formal approximation relations*

 $\triangleleft_{\tau} \subseteq \llbracket \tau \rrbracket \times \mathrm{PCF}_{\tau} \checkmark$

on types

are *logically* chosen to allow a proof by induction.



Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$\llbracket M \rrbracket \lhd_{\gamma} M \text{ implies } \underbrace{\forall V \left(\llbracket M \rrbracket = \llbracket V \rrbracket \implies M \Downarrow_{\gamma} V \right)}_{\text{adequacy}}$$

Definition of $d \triangleleft_{\gamma} M$ $(d \in [\![\gamma]\!], M \in \mathrm{PCF}_{\gamma})$ for $\gamma \in \{nat, bool\}$

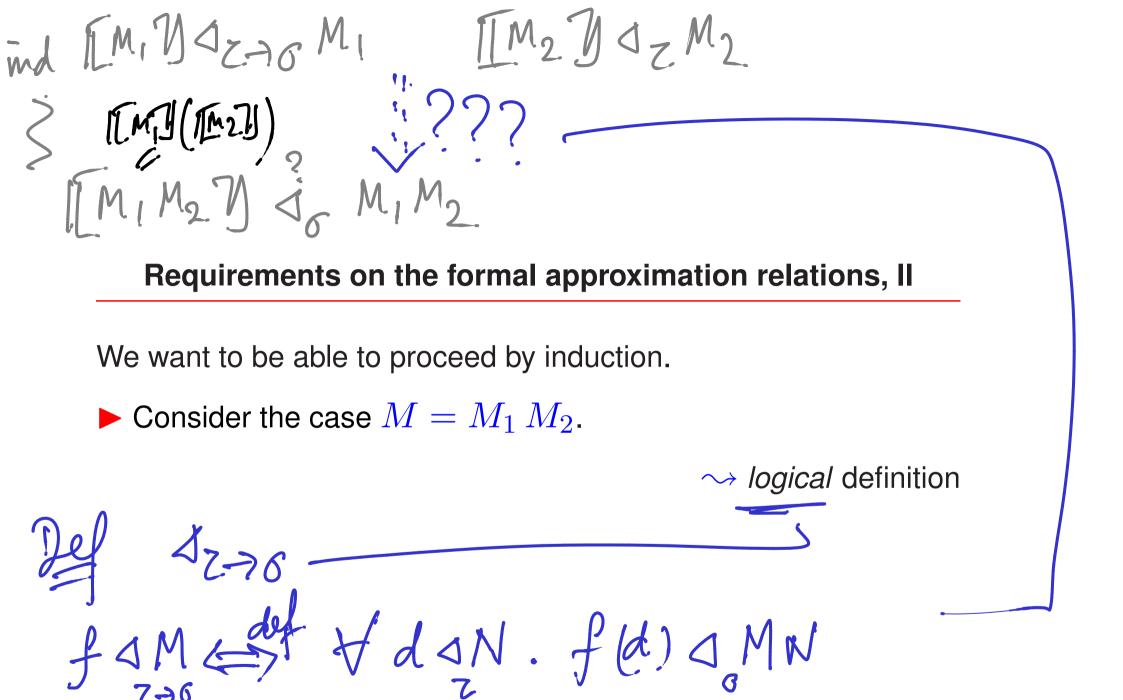
$$n \triangleleft_{nat} M \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \left(n \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \operatorname{succ}^{n}(\mathbf{0}) \right)$$

$$b \triangleleft_{bool} M \stackrel{\text{def}}{\Leftrightarrow} (b = true \Rightarrow M \Downarrow_{bool} \mathbf{true})$$
$$\& (b = false \Rightarrow M \Downarrow_{bool} \mathbf{false})$$

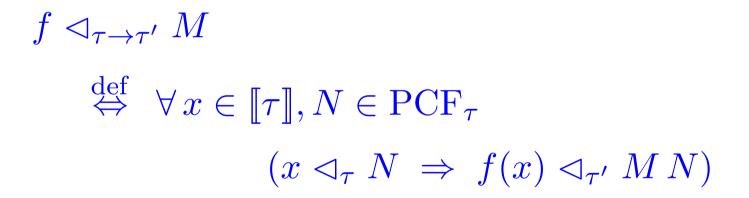
Proof of: $\llbracket M \rrbracket \lhd_{\gamma} M$ implies adequacy

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\begin{split} \mathbf{Case} \ \gamma &= nat. \\ \llbracket M \rrbracket = \llbracket V \rrbracket \\ &\implies \llbracket M \rrbracket = \llbracket \mathbf{succ}^n(\mathbf{0}) \rrbracket & \text{ for some } n \in \mathbb{N} \\ &\implies n = \llbracket M \rrbracket \triangleleft_{\gamma} M \\ &\implies M \Downarrow \mathbf{succ}^n(\mathbf{0}) & \text{ by definition of } \triangleleft_{nat} \end{split}
```

Case $\gamma = bool$ is similar.



Definition of
$$f \triangleleft_{\tau \to \tau'} M \ \left(f \in (\llbracket \tau \rrbracket \to \llbracket \tau' \rrbracket), M \in \mathrm{PCF}_{\tau \to \tau'} \right)$$



 $\begin{bmatrix} f_{\mathcal{H}}(M') & \mathcal{J}_{\mathcal{Z}} & f_{\mathcal{H}}(M') \\ \text{Requirements on the formal approximation relations, III} \end{bmatrix}$

We want to be able to proceed by induction.

• Consider the case $M = \mathbf{fix}(M')$.

→ *admissibility* property

> 2. property of a fixed point: Do it by



Admissibility property

Lemma. For all types τ and $M \in \mathrm{PCF}_{\tau}$, the set $\{ d \in \llbracket \tau \rrbracket \mid d \lhd_{\tau} M \}$

is an admissible subset of $[\tau]$.

Further properties

Lemma. For all types τ , elements $d, d' \in \llbracket \tau \rrbracket$, and terms $M, N, V \in \text{PCF}_{\tau}$,

1. If $d \sqsubseteq d'$ and $d' \triangleleft_{\tau} M$ then $d \triangleleft_{\tau} M$. (2) If $d \triangleleft_{\tau} M$ and $\forall V (M \Downarrow_{\tau} V \Longrightarrow N \Downarrow_{\tau} V)$ then $d \triangleleft_{\tau} N$.

Jd. [M'] [ZH)d] [[fnzn'y] Jzro fnz. M' Requirements on the formal approximation relations, IV We want to be able to proceed by induction. Consider the case $M = \mathbf{fn} \, x : \tau \, . \, M'$. \rightarrow substitutivity property for open terms (by def It would be enough to show M (by previous) [[M'] (a wid) & M'[N/x]

Fundamental property

Theorem. For all $\Gamma = \langle x_1 \mapsto \tau_1, \dots, x_n \mapsto \tau_n \rangle$ and all $\Gamma \vdash M : \tau$, if $d_1 \triangleleft_{\tau_1} M_1, \dots, d_n \triangleleft_{\tau_n} M_n$ then $\llbracket \Gamma \vdash M \rrbracket [x_1 \mapsto d_1, \dots, x_n \mapsto d_n] \triangleleft_{\tau} M [M_1/x_1, \dots, M_n/x_n]$.

NB. The case $\Gamma = \emptyset$ reduces to

 $\llbracket M \rrbracket \lhd_{\tau} M$

for all $M \in \mathrm{PCF}_{\tau}$.

Fundamental property of the relations \triangleleft_{τ}

Proposition. If $\Gamma \vdash M : \tau$ is a valid PCF typing, then for all Γ -environments ρ and all Γ -substitutions σ

 $\rho \triangleleft_{\Gamma} \sigma \; \Rightarrow \; \llbracket \Gamma \vdash M \rrbracket(\rho) \triangleleft_{\tau} M[\sigma]$

- $\rho \triangleleft_{\Gamma} \sigma$ means that $\rho(x) \triangleleft_{\Gamma(x)} \sigma(x)$ holds for each $x \in dom(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for x in M, each $x \in dom(\Gamma)$.