Denotational semantics of PCF terms, IV

$$
\begin{aligned}
&[\![\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M]\!](\rho) \\
&\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!] \,.\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d])
\end{aligned}
\qquad (x \notin dom(\Gamma))
$$

NB: $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping x to $d \in [\![\tau]\!]$ and otherwise acting like ρ.

(*) $f : D_1 \times D_2 \to D$ cont.

$\Downarrow$ claim

$\hat{f} : D_1 \to (D_2 \to D)$ cont.

$\hat{f}(d_1) = \lambda d. f(d_1, d)$

Assume (*)

- $\hat{f}$ is monotone.

$$d_1 \sqsubseteq d_1' \overset{?}{\Rightarrow} \hat{f}(d_1) \sqsubseteq \hat{f}(d_1')$$

$$\Downarrow$$

$$(d_1, d) \sqsubseteq (d_1', d)$$

f mon.

iff $\hat{f}(d_1)(d) \sqsubseteq \hat{f}(d_1')(d) \ \forall d$

$\hat{f}(d_1)(d) = f(d_1, d)$, $\quad \hat{f}(d_1')(d) = f(d_1', d)$

- $\hat{f}$ preserves lubs

$$\hat{f}\left(\bigsqcup_n d_n\right) \stackrel{?}{=} \bigsqcup_n \hat{f}(d_n)$$

iff $\forall d.$ $\hat{f}\left(\bigsqcup_n d_n\right)(d) \stackrel{?}{=} \left(\bigsqcup_n \hat{f}(d_n)\right)(d)$

$$\hat{f}\left(\bigsqcup_n d_n\right)(d) = f\left(\bigsqcup_n d_n, d\right)$$

$$\left(\bigsqcup_n \hat{f}(d_n)\right)(d) = \bigsqcup_n \left(\hat{f}(d_n)(d)\right) = \bigsqcup_n f(d_n, d)$$

$$f\left(\bigsqcup_n d_n, d\right) = \bigsqcup_n f(d_n, d)$$

f cont (in first arg.)

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{ \perp \}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

$$\textit{if} \quad [\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\textit{then} \quad [\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M]]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$$

- $M = \underline{fix}(M')$

$$\frac{M'(\underline{fix}\, M') \Downarrow V}{\underline{fix}(M') \Downarrow V}$$

by ind $[\![M'(\underline{fix}\, M')]\!] = [\![V]\!]$

$= [\![M']\!]([\![\underline{fix}\, M']\!])$

$= [\![M']\!](fix([\![M']\!]))$

$= fix\, [\![M']\!]$

$= [\![\underline{fix}\, M']\!]$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

if $M \Downarrow_\tau V$ *then* $[\![M]\!] = [\![V]\!] \in [\![\tau]\!]$.

- $M = M_1\, M_2$

$$\frac{M_1 \Downarrow \mathrm{fn}\, x.\, M \qquad M[M_2/x] \Downarrow V}{M_1\, M_2 \Downarrow V}$$

by ind. $[\![M_1]\!] = [\![\mathrm{fn}\, x.\, M]\!] = \lambda d.\, [\![M]\!][x \mapsto d]$

$[\![M[M_2/x]]\!] = [\![V]\!]$

Lemma

$=$

RTP: $[\![M_1\, M_2]\!] \overset{?}{=} [\![V]\!]$

$\|$

$[\![M_1]\!]([\![M_2]\!]) = [\![M]\!][x \mapsto [\![M_2]\!]]$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*.*

Then,

$$\begin{aligned} & [\![\Gamma \vdash M'[M/x]]\!](\rho) \\ & \quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big) \end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$*.*

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

Topic 7

Relating Denotational and Operational Semantics

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V \ .$$

NB. Adequacy does not hold at function types:

$$[\![\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x \not\Downarrow_{\tau\to\tau} \mathbf{fn}\, x : \tau.\, x$$

Adequacy proof idea

$[\![M]\!] = [\![V]\!] \Rightarrow M \Downarrow_{\gamma} V$

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ▶ Consider M to be $M_1\ M_2$, $\mathbf{fix}(M')$.

- $M = M_1\ M_2$

a subterm of M

not a subterm of M

$$\frac{M_1 \Downarrow fn\ x.M \qquad M[M_2/x] \Downarrow V}{M_1\ M_2 \Downarrow V}$$

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider M to be $M_1\,M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   for τ a ground type will imply adequacy

   This statement roughly takes the form:

$$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

$$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   by ind on types

   are *logically* chosen to allow a proof by induction.

• $\gamma = bool$

$true \overset{\in \mathbb{B}_\perp}{} \lhd_{bool} M \overset{def}{\iff} M \Downarrow_{bool} \underline{\underline{true}}$

$false \lhd_{bool} M \overset{def}{\iff} M \Downarrow_{bool} \underline{\underline{false}}$

Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall V \, ([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

Definition of $d \lhd_\gamma M$ $(d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$ for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \quad \overset{\text{def}}{\Leftrightarrow} \quad \big(n \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0})\big)$$

$$b \lhd_{bool} M \quad \overset{\text{def}}{\Leftrightarrow} \quad (b = \mathit{true} \Rightarrow M \Downarrow_{bool} \mathbf{true})$$
$$\& \ (b = \mathit{false} \Rightarrow M \Downarrow_{bool} \mathbf{false})$$

Proof of: $[\![M]\!] \lhd_\gamma M$ implies adequacy

Case $\gamma = nat$.

$$
\begin{aligned}
[\![M]\!] &= [\![V]\!] \\
&\Longrightarrow\ [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] && \text{for some } n \in \mathbb{N} \\
&\Longrightarrow\ n = [\![M]\!] \lhd_\gamma M \\
&\Longrightarrow\ M \Downarrow \mathbf{succ}^n(\mathbf{0}) && \text{by definition of } \lhd_{nat}
\end{aligned}
$$

Case $\gamma = bool$ is similar.

ind $[\![M_1]\!] \lhd_{\tau\to\sigma} M_1$ $\quad [\![M_2]\!] \lhd_\tau M_2$

$\Downarrow$???

$[\![M_1]\!]([\![M_2]\!])$

$[\![M_1 M_2]\!] \overset{?}{\lhd}_\sigma M_1 M_2$

Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$\rightsquigarrow$ *logical* definition

Def $\lhd_{\tau\to\sigma}$

$f \lhd_{\tau\to\sigma} M \overset{def}{\iff} \forall d \lhd_\tau N .\ f(d) \lhd_\sigma M N$

Definition of
$f \lhd_{\tau\to\tau'} M \;\left(f \in (\llbracket\tau\rrbracket \to \llbracket\tau'\rrbracket), M \in \mathrm{PCF}_{\tau\to\tau'}\right)$

$$
\begin{aligned}
& f \lhd_{\tau\to\tau'} M \\
& \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\, x \in \llbracket\tau\rrbracket, N \in \mathrm{PCF}_{\tau} \\
& \qquad\qquad\qquad (x \lhd_{\tau} N \;\Rightarrow\; f(x) \lhd_{\tau'} M\,N)
\end{aligned}
$$

$\mathrm{fix}(\llbracket M' \rrbracket)$

$\|$

$\llbracket \underline{\mathrm{fix}}(M') \rrbracket \lhd_\tau \underline{\mathrm{fix}}(M')$

a property of a fixed point: Do it by Scott Ind.

Requirements on the formal approximation relations, III

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fix}(M')$.

Need

$\rightsquigarrow$ *admissibility* property

Admissibility property

Lemma. *For all types* τ *and* $M \in \mathrm{PCF}_\tau$, *the set*

$$\{\, d \in [\![\tau]\!] \mid d \lhd_\tau M \,\}$$

is an admissible subset of $[\![\tau]\!]$.

Further properties

Lemma. *For all types* τ, *elements* $d, d' \in [\![\tau]\!]$, *and terms* $M, N, V \in \mathrm{PCF}_\tau$,

1. *If* $d \sqsubseteq d'$ *and* $d' \lhd_\tau M$ *then* $d \lhd_\tau M$.
2. *If* $d \lhd_\tau M$ *and* $\forall V\,(M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ *then* $d \lhd_\tau N$.

$\lambda d.\, [\![M']\!][x \mapsto d]$

$\shortparallel$

$[\![\mathrm{fn}\, x.\, M']\!] \lhd^{?}_{\tau\to\sigma} \mathrm{fn}\, x.\, M'$

Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fn}\, x : \tau\, .\, M'$.

(by def)

$\rightsquigarrow$ *substitutivity* property for open terms

$\text{iff } \forall d \lhd_\tau N.\ [\![M']\!][x \mapsto d] \lhd_\sigma (\mathrm{fn}\, x.\, M')(N)$

It would be enough to show $\Uparrow$

(by previous lemma) $[\![M']\!][x \mapsto d] \lhd_\sigma M'[N/x]$

Fundamental property

Theorem. *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$.

NB. The case $\Gamma = \emptyset$ reduces to

$$[\![M]\!] \lhd_\tau M$$

for all $M \in \mathrm{PCF}_\tau$.

Fundamental property of the relations $\lhd_\tau$

Proposition. *If $\Gamma \vdash M : \tau$ is a valid PCF typing, then for all Γ-environments ρ and all Γ-substitutions σ*

$$\rho \lhd_\Gamma \sigma \;\Rightarrow\; [\![\Gamma \vdash M]\!](\rho) \lhd_\tau M[\sigma]$$

- $\rho \lhd_\Gamma \sigma$ means that $\rho(x) \lhd_{\Gamma(x)} \sigma(x)$ holds for each $x \in dom(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for x in M, each $x \in dom(\Gamma)$.