Contextual equivalence of PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \cong_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V : \gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \ \Leftrightarrow\ \mathcal{C}[M_2] \Downarrow_\gamma V.$$

PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[\![\tau]\!]$.
- Closed PCF terms $M : \tau \mapsto$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.

NB: In defining $[\![\mathbf{fn}\, x.M]\!] : [\![\tau]\!] \to [\![\sigma]\!]$ we need define $[\![x : \tau \vdash M : \sigma]\!]$.

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- Closed PCF terms $M : \tau \quad \mapsto \quad$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.
- Compositionality.

  In particular: $[\![M]\!] = [\![M']\!] \;\Rightarrow\; [\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[M']]\!]$.
- Soundness.

  For any type τ, $M \Downarrow_\tau V \;\Rightarrow\; [\![M]\!] = [\![V]\!]$.
- Adequacy.

  For $\tau = \mathit{bool}$ or nat, $[\![M]\!] = [\![V]\!] \in [\![\tau]\!] \implies M \Downarrow_\tau V$.

Theorem. *For all types* τ *and closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$, *if* $[\![M_1]\!]$ *and* $[\![M_2]\!]$ *are equal elements of the domain* $[\![\tau]\!]$*, then* $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.

$[\![M_1]\!] = [\![M_2]\!]$ in $[\![\tau]\!]$ $(*)$

$\Rightarrow M_1 \cong_{ctx} M_2 : \tau$

γ ground type

Proof: Assume $(*)$.

$\mathcal{C}[M_1] \Downarrow_\gamma V \Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!]$ soundness

$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!]$ comp.

$\Rightarrow \mathcal{C}[M_2] \Downarrow_\gamma V$. adequacy $\boxtimes$

Theorem. *For all types τ and closed terms $M_1, M_2 \in \mathrm{PCF}_\tau$, if $[\![M_1]\!]$ and $[\![M_2]\!]$ are equal elements of the domain $[\![\tau]\!]$, then $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.*

Proof.

$$\mathcal{C}[M_1] \Downarrow_{nat} V \Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!] \quad \text{(soundness)}$$

$$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!] \quad \text{(compositionality on } [\![M_1]\!] = [\![M_2]\!]\text{)}$$

$$\Rightarrow \mathcal{C}[M_2] \Downarrow_{nat} V \quad \text{(adequacy)}$$

and symmetrically. □

Proof principle

To prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!]$$

$\boxed{?}$ The proof principle is sound, but is it complete? That is, is equality in the denotational model also a necessary condition for contextual equivalence?

Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

by induction on the derivation

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

by ind on the structure of types

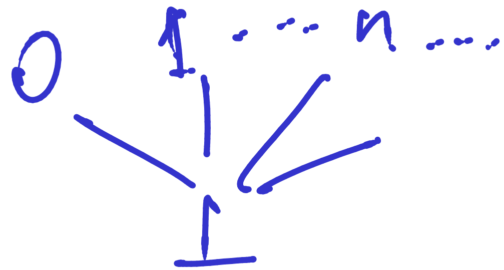
between domains.

$\Gamma \equiv \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$

$[\![\Gamma]\!]$ compositionally uses $[\![\tau_i]\!]$

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp \qquad \text{(flat domain)}$$



$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp \qquad \text{(flat domain)}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF type environments

$$\llbracket \Gamma \rrbracket \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} \llbracket \Gamma(x) \rrbracket \qquad (\Gamma\text{-environments})$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in \llbracket \Gamma(x) \rrbracket$ for all $x \in dom(\Gamma)$

Example:

The flat domain over the empty set

1. For the empty type environment $\emptyset$,

$$\llbracket \emptyset \rrbracket = \{ \perp \} = \emptyset_\perp$$

where $\perp$ denotes the unique partial function with $dom(\perp) = \emptyset$.

singleton set consisting of x

2. $$[\![\langle x \mapsto \tau \rangle]\!] = (\{x\} \to [\![\tau]\!])$$

bijection

2. $[\![\langle x \mapsto \tau \rangle]\!] = \left(\{\, x \,\} \to [\![\tau]\!]\right) \cong [\![\tau]\!]$

(*) in functional notation $\rho(x_i)$ is the value of x_i in the environment ρ.

2. $[\![\langle x \mapsto \tau \rangle]\!] = (\{\, x \,\} \to [\![\tau]\!]) \cong [\![\tau]\!]$

3.

$$
\begin{aligned}
& [\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
& \quad \cong (\{\, x_1 \,\} \to [\![\tau_1]\!]) \times \ldots \times (\{\, x_n \,\} \to [\![\tau_n]\!]) \\
& \quad \cong [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

Idea: $\rho \in [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$ is an "environment".

$\rho = (d_1, d_2, \ldots, d_n)$

d_i is the value of x_i in the environment. (*)

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash 0 : nat]\!] : [\![\Gamma]\!] \to [\![nat]\!]$$

$$\| \; \lambda \rho \in [\![\Gamma]\!] . 0$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

Denotational semantics of PCF terms, I

$$\frac{x \in dom(\Gamma) \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau}$$

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

in other terms

$$\frac{i = 1 \ldots n}{x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i}$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} \mathit{true} \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} \mathit{false} \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad (x \in dom(\Gamma))$$

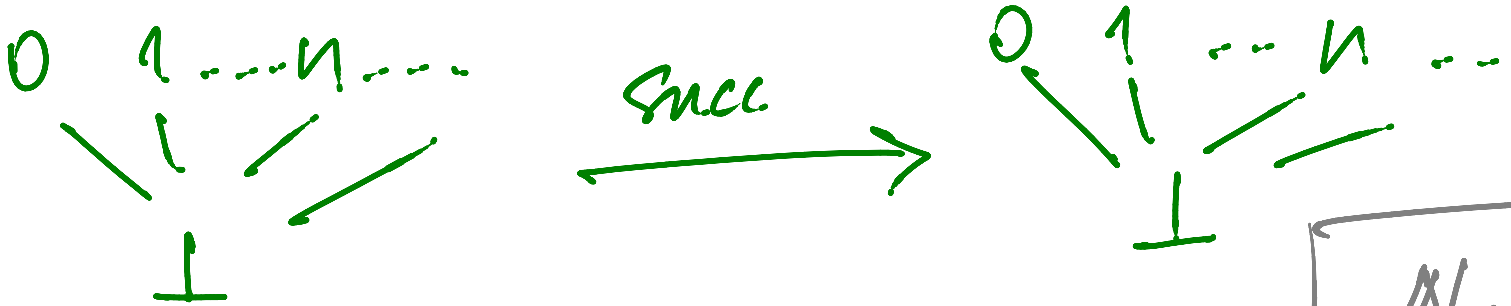
in other terms

$$[\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i]\!](d_1, \ldots, d_n) = d_i$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \perp \\ \perp & \text{if } [\![\Gamma \vdash M]\!](\rho) = \perp \end{cases}$$



$$\perp \longmapsto \perp$$

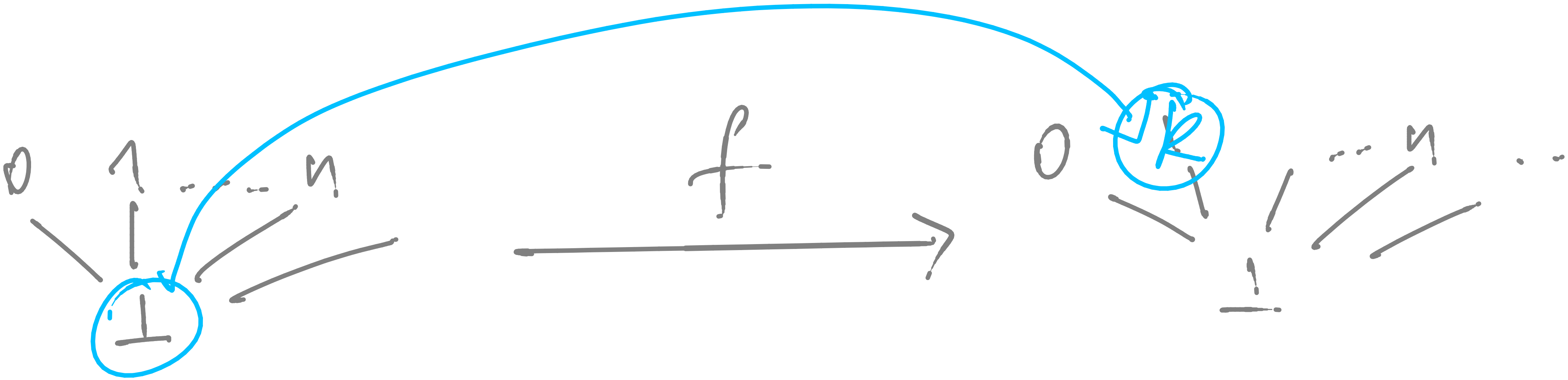
$$\mathbb{N} \ni n \longmapsto n+1$$

$\mathbb{N}_\perp \xrightarrow{f} \mathbb{N}_\perp$

continuous

if $f(\perp) = k \in \mathbb{N}$

then $\forall n . f(n) = k$



$$\forall i\ \bot \sqsubseteq i \underset{\text{mon.}}{\Rightarrow} \underbrace{f(\bot)}_{= k} \sqsubseteq f(i) \Rightarrow f(i) = k$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

$pred : \mathbb{N}_\bot \to \mathbb{N}_\bot$ cont.

$$\begin{cases} \bot, 0 \longmapsto \bot \\ n \geqslant 1 \longmapsto n - 1 \end{cases}$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$\frac{\Gamma \vdash M_1 : \tau \to \sigma \qquad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \sigma}$$

Denotational semantics of PCF terms, III

$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\sigma]\!])$

$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

$$[\![\Gamma \vdash \textbf{if } M_1 \textbf{ then } M_2 \textbf{ else } M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash M_1\, M_2]\!](\underbrace{\rho}_{\in [\![\Gamma]\!]}) \stackrel{\text{def}}{=} (\underbrace{[\![\Gamma \vdash M_1]\!](\rho)}_{\in [\![\tau]\!] \to [\![\sigma]\!]})\, (\underbrace{[\![\Gamma \vdash M_2]\!](\rho)}_{\in [\![\tau]\!]})$$

$$\frac{\Gamma, x:\tau \vdash M:\sigma}{\Gamma \vdash \mathbf{fn}\, x:\tau\, .\, M : \tau \to \sigma}$$

Denotational semantics of PCF terms, IV

$$[\![\Gamma, x:\tau \vdash M]\!] : [\![\Gamma, x:\tau]\!] \to [\![\sigma]\!]$$

$$[\![\Gamma \vdash \mathbf{fn}\, x : \tau\, .\, M]\!](\rho)$$

$$\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!]\, .\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d])$$

$(x \notin dom(\Gamma))$

$\cong$

$[\![\Gamma]\!] \times [\![\tau]\!]$

curying (is continuous)

NB: $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping x to $d \in [\![\tau]\!]$ and otherwise acting like ρ.

$$[\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\sigma]\!])$$