Modeling Data Types

α, β types | D, E domains

$\alpha * \beta$ type | $D \times E$ domain

$\alpha \to \beta$ type | $(D \to E)$ domain

Continuous functions of two arguments

Proposition. *Let D, E, F be cpo's. A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$

$$f(d\,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$f : D \times E \to F$

by Def.

monotone $(d, e) \sqsubseteq (d', e') \Rightarrow f(d, e) \sqsubseteq f(d', e')$

cont. $f\left(\bigsqcup_n (d_n, e_n)\right) = \bigsqcup_n f(d_n, e_n)$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x, y) \sqsubseteq f(x', y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)}$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

has lubs of countable chains

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad \sqsubseteq \bigsqcup_n f_n \quad$ in $(D \to E)$

$(n \in \mathbb{N})$

Def: $\left(\bigsqcup_n f_n\right)(d) = \bigsqcup_n \left(f_n(d)\right)$

Check $\bigsqcup_n f_n$ is

- monotone: $d \sqsubseteq d' \overset{?}{\Longrightarrow} (\bigsqcup_n f_n)(d) \sqsubseteq (\bigsqcup_n f_n)(d')$

$$(\bigsqcup_n f_n)(d) \overset{\text{def}}{=} \bigsqcup_n (f_n(d)) \qquad (\bigsqcup_n f_n)(d') \overset{\text{def}}{=} \bigsqcup_n (f_n(d'))$$

f_n mon..

$$d \sqsubseteq d' \Longrightarrow f_n(d) \sqsubseteq f_n(d') \overset{\text{lemma}}{\Longrightarrow} \bigsqcup_n (f_n(d)) \sqsubseteq \bigsqcup_n (f_n(d'))$$

- $\bigsqcup_n f_n$ preserves lubs.

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k d_k\right) \overset{?}{=} \bigsqcup_k \left(\bigsqcup_n f_n\right)(d_k)$$

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k d_k\right) \overset{\text{def}}{=} \bigsqcup_n f_n\left(\bigsqcup_k d_k\right) \overset{\text{cont.}}{=} \bigsqcup_n \bigsqcup_k f_n(d_k) = \bigsqcup_i f_i(d_i)$$

$$\bigsqcup_k \left(\bigsqcup_n f_n\right)(d_k) \overset{\text{def}}{=} \bigsqcup_k \bigsqcup_n f_n(d_k) \overset{\text{lemma}}{=} \bigsqcup_i f_i(d_i)$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D\to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n = \lambda d \in D. \bigsqcup_{n\geq 0} f_n(d) \ .$$

$\perp_{(D\to E)} = \lambda d \in D.\ \perp_E$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d) \ .$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D \to E}(d) = \perp_E$, all $d \in D$.

ML: fn $(g, f) \Rightarrow$ fn $d \Rightarrow g(f(d))$

Continuity of composition

Exercise

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \ = \ \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

a least pre fixed point, calculated as $\mathit{fix}(f) = \bigcup_n f^n(\bot)$

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

- **fix is monotone:**

$$f \sqsubseteq g \overset{?}{\Rightarrow} \underline{fix}(f) \sqsubseteq \underline{fix}(g)$$

Assume

$$f \sqsubseteq g$$

$$\Downarrow$$

$$\underbrace{f(\text{fix}\, g) \sqsubseteq g(\text{fix}\, g) \sqsubseteq \text{fix}\, g}$$

$$\frac{f(\text{fix}\, g) \sqsubseteq \text{fix}\, g}{\underline{fix}(f) \sqsubseteq \underline{fix}(g)}$$

- <u>$\underline{fix}$ is cont:</u>

$$\underline{fix}\left(\bigsqcup_n f_n\right) = \bigsqcup_n \underline{fix}(f_n)$$

$$\bigsqcup_n \bigsqcup_k f_n(\underline{fix}\, f_k) = \bigsqcup_i f_i(\underline{fix}\, f_i)$$

$$=$$

$$\bigsqcup_n f_n\left(\bigsqcup_k \underline{fix}\, f_k\right)$$

$$=$$

$$\frac{\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k \underline{fix}(f_k)\right) \sqsubseteq \bigsqcup_k \underline{fix}(f_k)}{\underline{fix}\left(\bigsqcup_n f_n\right) \sqsubseteq \bigsqcup_n \underline{fix}(f_n)}$$

$$\dfrac{\forall n \qquad \dfrac{\dfrac{\checkmark}{f_n \sqsubseteq \bigsqcup_k f_k}}{\underline{fix}(f_n) \sqsubseteq \underline{fix}\left(\bigsqcup_k f_k\right)}\ (\text{fix mon})}{\bigsqcup_n \underline{fix}(f_n) \sqsubseteq \underline{fix}\left(\bigsqcup_n f_n\right)}$$

Topic 4

Scott Induction

NICE PROPERTY

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$fix(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

$$\frac{\forall d \quad d \in S \Rightarrow f(d) \in S \ (*)}{fix(f) \in S}$$

$fix(f) = \bigsqcup_n f^n(\perp)$

(NICE 1) $\perp \in S$

$\Downarrow *$

$f(\perp) \in S$

$\Downarrow$

$f^2(\perp) \in S$

$\Downarrow$

$f^n(\perp) \in S \ \forall n$

(NICE 2) ...

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

(NRE$_2$) $$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n\geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\bot \in S$.

$$\frac{\forall d \in D.\ d \in S \Rightarrow f(d) \in S}{fix(f) \in S} \qquad S \text{ admissible}$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n \geq 0} d_n\right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

  of D is chain-closed.

- The subsets

$$\{(x,y) \in D \times D \mid x \sqsubseteq y\}$$

  and

$$\{(x,y) \in D \times D \mid x = y\}$$

  of $D \times D$ are chain-closed.

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\ f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

$$\mathit{fix} \sqsubseteq f\,d \sqsubseteq d$$

mon.

$$x \sqsubseteq d \overset{?}{\Rightarrow} f x \sqsubseteq d$$

equiv.

$$x \in {\downarrow}(d) \Rightarrow f(x) \in {\downarrow}(d)$$

$$\underline{\mathit{fix}}(f) \sqsubseteq d \iff \underline{\mathit{fix}}(f) \in {\downarrow}(d)$$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in {\downarrow}(d) \quad &\implies \quad x \sqsubseteq d \\ &\implies \quad f(x) \sqsubseteq f(d) \\ &\implies \quad f(x) \sqsubseteq d \\ &\implies \quad f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d) \ .$$

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

$S = \{x \mid fx \sqsubseteq gx\}$ Show $\mathit{fix}(g) \in S$

$fgx \sqsubseteq gfx \sqsubseteq ggx$

$$\frac{\dfrac{fx \sqsubseteq gx \overset{?}{\Rightarrow} f(gx) \sqsubseteq g(gx)}{x \in S \Rightarrow g(x) \in S}}{\mathit{fix}(g) \in S}$$

$\Downarrow$

$f(\mathit{fix}\,g) \sqsubseteq g(\mathit{fix}\,g) = \mathit{fix}(g)$

$\Downarrow$

$\mathit{fix}(f) \sqsubseteq \mathit{fix}(g)$

Exercise:

$$fg \sqsubseteq gf \wedge f(\bot) \sqsubseteq g(\bot)$$

$$\Rightarrow \bigsqcup_n f^n(\bot) \sqsubseteq \bigsqcup_n g^n(\bot)$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$