Lemma. *Let* D *be a cpo. Suppose that the doubly-indexed family of elements* $d_{m,n} \in D$ *(*$m, n \geq 0$*) satisfies*

$$m \leq m' \ \& \ n \leq n' \ \Rightarrow \ d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \ \sqsubseteq \ \bigsqcup_{n \geq 0} d_{1,n} \ \sqsubseteq \ \bigsqcup_{n \geq 0} d_{2,n} \ \sqsubseteq \ \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \ \sqsubseteq \ \bigsqcup_{m \geq 0} d_{m,1} \ \sqsubseteq \ \bigsqcup_{m \geq 0} d_{m,3} \ \sqsubseteq \ \ldots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) \ .$$

$$\overline{\forall i\ x_i \sqsubseteq \bigsqcup_i x_i}$$

$$\frac{\forall i.\ x_i \sqsubseteq y}{\bigsqcup_i x_i \sqsubseteq y}$$

$$\left.\begin{array}{l} \forall m\ \left(\bigsqcup_n d_{m,n}\right) \sqsubseteq \bigsqcup_m \left(\bigsqcup_n d_{m,n}\right) \\ \forall n\ d_{m,n} \sqsubseteq \bigsqcup_n d_{m,n} \end{array}\right\} \Rightarrow \begin{array}{l} \forall m \forall n \\ d_{m,n} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n} \end{array}$$

$$\Downarrow$$

Exercise

$$\frac{}{\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}} \qquad \frac{\forall k\ d_{kk} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}{\bigsqcup_k d_{kk} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}$$

$$\bigsqcup_m \left(\bigsqcup_n d_{m,n}\right) = \bigsqcup_k d_{k,k}$$

NB: computable $\Rightarrow$ continuous.

NB: $f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n f(d_n)$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  1. it is monotone, and

  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

Iff

$$f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n f(d_n)$$

$\bigsqcup_n d_n$

$\vdots$

$\sqcup\!\mid$

d_n

$\vdots$

$\sqcup\!\mid$

d_0

$f(\bigsqcup_n d_n)$

$\sqcup\!\mid$

$\bigsqcup_n f(d_n)$

$\vdots$

$f(d_n)$

$\vdots$

$\sqcup\!\mid$

$f(d_1)$

$\sqcup\!\mid$

$f(d_0)$

$$D \xrightarrow{f} E$$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f\Big(\bigsqcup_{n\geq 0} d_n\Big) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\bot) = \bot$.

$\perp \sqsubseteq f(\perp) \Rightarrow f(\perp) \sqsubseteq f(f(\perp))$

$\Rightarrow f^2(\perp) \sqsubseteq f^3(\perp)$

recall, is a fixed point.

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f(\mathit{fix}(f)) = \mathit{fix}(f)$, and hence is the least fixed point of f.

$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots \sqsubseteq \bigsqcup_n f^n(\perp)$

$\underline{\mathrm{fix}}(f) = \bigsqcup_n f^n(\bot)$ is a least pre-fixed point

(1) It is a prefixed point; that is,

$$f(\underline{\mathrm{fix}}\, f) \sqsubseteq \underline{\mathrm{fix}}\, f$$

equiv.

$$f\left(\bigsqcup_n f^n(\bot)\right) \sqsubseteq \bigsqcup_n f^n(\bot)$$

$$f\bot \sqsubseteq f f\bot \sqsubseteq f f^2(\bot) \sqsubseteq \cdots \sqsubseteq f f^n(\bot) \sqsubseteq \cdots \sqsubseteq f\left(\bigsqcup_n f^n(\bot)\right)$$

$$f\left(\bigsqcup_n f^n \bot\right) \underset{\text{cont}}{=} \bigsqcup_{n\geq 0} f(f^n \bot) = \bigsqcup_{k\geq 1} f^k(\bot)$$

$$= \bigsqcup_{k\geq 0} f^k(\bot)$$

(2) Show $fix(f)$ is least amongst all prefixed points

So consider x s.t. $f(x) \sqsubseteq x$.

<u>RTP</u> $fix(f) \sqsubseteq x$

equiv. $\bigsqcup_n f^n(\bot) \sqsubseteq x$

$n=0$: $\bot \sqsubseteq x$ ✓ $\Rightarrow f(\bot) \sqsubseteq f(x) \sqsubseteq x$

$n=1$: $f(\bot) \sqsubseteq x$? ✓

by induction on n

$$\frac{\forall n. \quad f^n(\bot) \sqsubseteq x}{\bigsqcup_n f^n(\bot) \sqsubseteq x}$$

$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$
\begin{aligned}
& [\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] \\
&= \mathit{fix}(f_{[\![B]\!],[\![C]\!]}) \\
&= \textstyle\bigsqcup_{n \geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot) \\
&= \ \lambda s \in \mathit{State}. \\
&\quad \begin{cases} [\![C]\!]^{k}(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ & \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}
\end{aligned}
$$

$f_{[\![B]\!],[\![C]\!]} = \lambda w.\, \lambda s.$
$\mathrm{if}([\![B]\!]s, w([\![C]\!]s), s)$

$\bot$ = state transformer
with empty graph.

Topic 3

Constructions on Domains

In ML new types from old

α, β types

PRODUCTS $\alpha * \beta$ types

FUNCTIONS $\alpha \to \beta$

BUILT-IN TYPES

int,

bool, ...

DATATYPES

ENUMERATED

$\cap$|

INDUCTIVE (lists, trees, ...)

$\cap$|

RECURSIVE

Discrete cpo's and flat domains

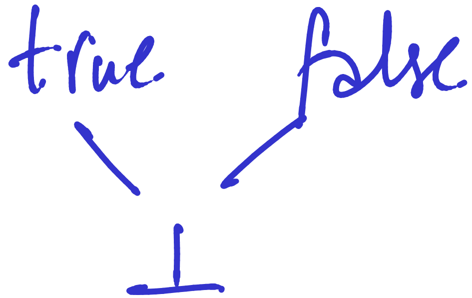
model built-in datatypes

For any set X, the relation of equality

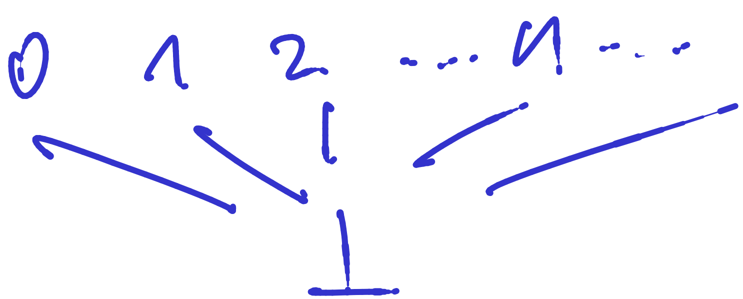
$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

$\mathbb{B} = \{true, false\} \rightsquigarrow \mathbb{B}_\perp =$



$\mathbb{N} = \{0, 1, \dots\} \rightsquigarrow \mathbb{N}_\perp =$



Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Product D_1, D_2 domains

$D_1 \times D_2$ models $\alpha * \beta$ in ML.

- the underlying set
  $$\{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$
- the partial order
  $$(d_1, d_2) \sqsubseteq_{D_1 \times D_2} (d_1', d_2') \text{ iff } d_1 \sqsubseteq_{D_1} d_1' \wedge d_2 \sqsubseteq_{D_2} d_2'$$
- The least is: $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$

• it is complete: consider a chain in $D_1 \times D_2$

$$(d_0, e_0) \sqsubseteq (d_1, e_1) \sqsubseteq \cdots \sqsubseteq (d_n, e_n) \sqsubseteq \cdots$$

need to find $\bigsqcup_n (d_n, e_n)$?

// def'n gives lubs.

$$\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right)$$

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ in D_1 $\qquad \bigsqcup_n d_n \in D_1$

$e_0 \sqsubseteq e_1 \sqsubseteq \cdots \sqsubseteq e_n \sqsubseteq \cdots$ in D_2 $\qquad \bigsqcup_n e_n \in D_2$

RTP ① $\forall n.\ (d_n, e_n) \overset{?}{\sqsubseteq} \left(\bigsqcup_n d_n, \bigsqcup_n e_n\right)$

$\overset{?}{\iff}$

$\forall n.\ d_n \sqsubseteq \bigsqcup_n d_n \wedge e_n \sqsubseteq \bigsqcup_n e_n$ ✓

② Let $(d_n, e_n) \sqsubseteq (x, y)\ \forall n \implies \forall n\ \ d_n \sqsubseteq x$, $e_n \sqsubseteq y$

RTP $\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) \sqsubseteq (x, y)$

$\overset{?}{\iff}$

$\bigsqcup_n d_n \sqsubseteq x \wedge \bigsqcup_n e_n \sqsubseteq y.$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \ \& \ d_2 \sqsubseteq_2 d_2' \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{\overline{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}) \enspace .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1\times D_2} = (\perp_{D_1}, \perp_{D_2})$.