$$d = \bigsqcup_n d_n$$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

Thesis*

> All domains of computation are
> complete partial orders with a least element.

C

idea

$d = \bigsqcup_n d_n$

$\bigsqcup_n f(d_n) = f(d)$

Thesis*

All domains of computation are
complete partial orders with a least element.

All computable functions are
continuous.

(monotone +
preservation of $\bigsqcup_n$

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \ (n \in \mathbb{N}) \sqsubseteq \cdots \sqsubseteq d$$

$$\xrightarrow{\quad f \quad}$$

$$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots \sqsubseteq f(d)$$

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

d is an upper bound of the dn's

$$\forall d \in D \,.\, \boxed{(\forall m \geq 0 \,.\, d_m \sqsubseteq d)} \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

$\bigsqcup_n d_n$ is least among all upper bounds.

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

d, $\sqcup$!! d!, $\sqcup$!!, d_n, $\sqcup$!, d_1, $\sqcup$!, d_0

$$\frac{}{\bot \sqsubseteq x}$$

(lub1)

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

(lub2)

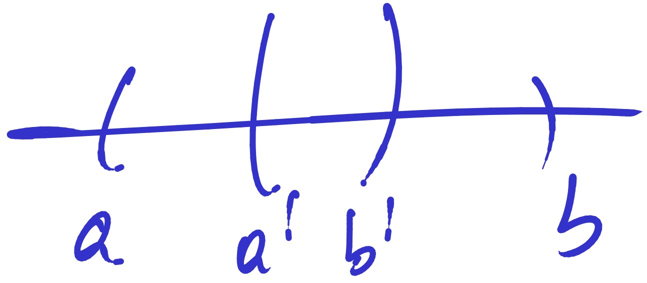
$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

$\mathbb{I}[0,1]$ ~ elements are open intervals in $[0,1]$

(a,b) $\quad 0 \le a < b \le 1$

$[a,b]$ $\quad 0 \le a \le b \le 1$

$(a,b) \sqsubseteq (a',b')$

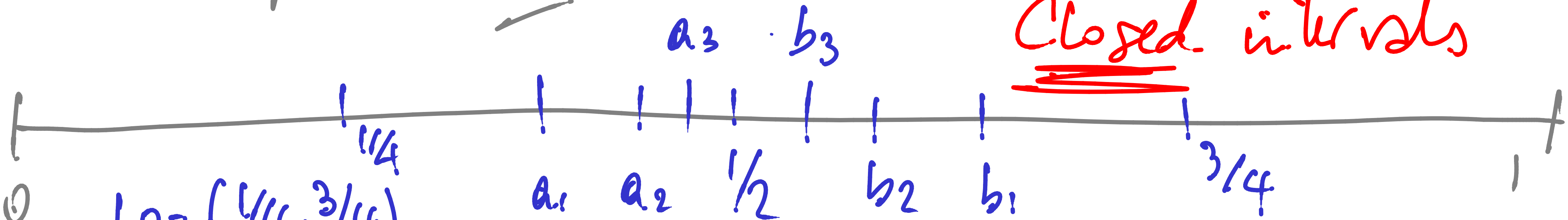


is it complete?

No!

we have a least elt $(0,1)$

But consider **closed** intervals



$i_0 = (1/4, 3/4)$

$i_1 = (a_1, b_1)$

$i_n = (a_n, b_n)$

$\sqcup_n i_n = [1/2, 1/2]$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

the $f_n \sqsubseteq g$

graph$(f_n) \subseteq$ graph(g)

$\sqcup$ / ?

graph (f) ?

yes !

$f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots \sqsubseteq f_n \sqsubseteq \ldots$

$\equiv$ graph$(f_0) \subseteq$ graph$(f_1) \subseteq \ldots \subseteq$ graph$(f_n) \subseteq \ldots$

$\bigcup_n$ graph(f_n) is a relation that is functional hence the graph of a partial function say f

Domain of partial functions, $X \rightharpoonup Y$

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Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\perp$ is the totally undefined partial function.

$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$ $\qquad d \sqsubseteq x \qquad d = \bigsqcup_n d$

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots \sqsubseteq d_N \sqsubseteq \ldots$

$d \sqsubseteq d \sqsubseteq d \sqsubseteq \ldots \sqsubseteq x$

$$\frac{\forall i.\ x_i \sqsubseteq x}{\bigsqcup_i x_i \sqsubseteq x}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{d_i's form a chain}}{d_n \sqsubseteq d_{N+n}} \qquad \dfrac{\forall i.\ d_{N+i} \sqsubseteq \bigsqcup_i d_{N+i}}{d_{N+n} \sqsubseteq \bigsqcup_i d_{N+i}}}{\forall n.\ d_n \sqsubseteq \bigsqcup_i d_{N+i}}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n d_{N+n}} \qquad \dfrac{\text{(exercise)} \\ \vdots}{\bigsqcup_n d_{N+n} \sqsubseteq \bigsqcup_n d_n}}{\bigsqcup_n d_n = \bigsqcup_n d_{N+n}}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\begin{array}{ccccccccc}
e_0 \sqsubseteq & e_1 \sqsubseteq & \cdots & \sqsubseteq e_n \sqsubseteq & \cdots & & & \sqsubseteq \bigsqcup_n e_n \\
\sqcup\!\!\!| & \sqcup\!\!\!| & \cdots & \sqcup\!\!\!| & \cdots & & & \sqcup\!\!\!| \; ? \\
d_0 \sqsubseteq & d_1 \sqsubseteq & \cdots & \sqsubseteq d_n \sqsubseteq & \cdots & & \cdots & \sqsubseteq \bigsqcup_n d_n
\end{array}$$

$$\dfrac{\forall n \qquad \dfrac{\dfrac{\checkmark}{d_n \sqsubseteq e_n} \qquad \dfrac{\checkmark}{e_n \sqsubseteq \bigsqcup_n e_n}}{d_n \sqsubseteq \bigsqcup_n e_n}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$\bigsqcup_m d_0^{(m)} \sqsubseteq \bigsqcup_m d_1^{(m)} \sqsubseteq \cdots \sqsubseteq \bigsqcup_n \left(\bigsqcup_m d_n^{(m)} \right) \overset{?}{=} \bigsqcup_m \left(\bigsqcup_n d_n^{(m)} \right)$$

$$\begin{array}{ccccccccccc}
\rotatebox{90}{$\sqsubseteq$} & & & & & & & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_0^{(m)} & \sqsubseteq & d_1^{(m)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(m)} & \sqsubseteq & \cdots \bigsqcup_k d_k^{(k)} & & \sqsubseteq \bigsqcup_n d_n^{(m)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & & & \vdots & & & & \\
d_0^{(2)} & \sqsubseteq & d_1^{(2)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(2)} & \sqsubseteq & \cdots & & \sqsubseteq \bigsqcup_n d_n^{(2)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_0^{(1)} & \sqsubseteq & d_1^{(1)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(1)} & \sqsubseteq & \cdots & & \sqsubseteq \bigsqcup_n d_n^{(1)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \cdots & & \rotatebox{90}{$\sqsubseteq$} & & \cdots & & \rotatebox{90}{$\sqsubseteq$} \\
d_0^{(0)} & \sqsubseteq & d_1^{(0)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(0)} & \sqsubseteq & \cdots & & \sqsubseteq \bigsqcup_n d_n^{(0)}
\end{array}$$

Fact:

$$
\begin{tikzcd}
\bigsqcup_m \bigsqcup_n d^{(m)}_n \arrow[dr, equal] & = & \bigsqcup_n \bigsqcup_m d^{(m)}_n \arrow[dl, equal] \\
& \bigsqcup_k d^{(k)}_k &
\end{tikzcd}
$$

Idea

$$\bigsqcup_m \bigsqcup_n d^{(m)}_n \overset{?}{\sqsubseteq} \bigsqcup_k d^{(k)}_k$$